UNIT - 5: Analog Filter Design

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Analog Filter Design:[1, 2, 3, 4]

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Unit 5: Analog Filter Design:

PART - B-Unit 5: Analog Filter Design:

- Characteristics of commonly used analog filters
- Butterworth and Chebyshev filters
- Analog to analog frequency transformations.

Magnitude Characteristic of filters

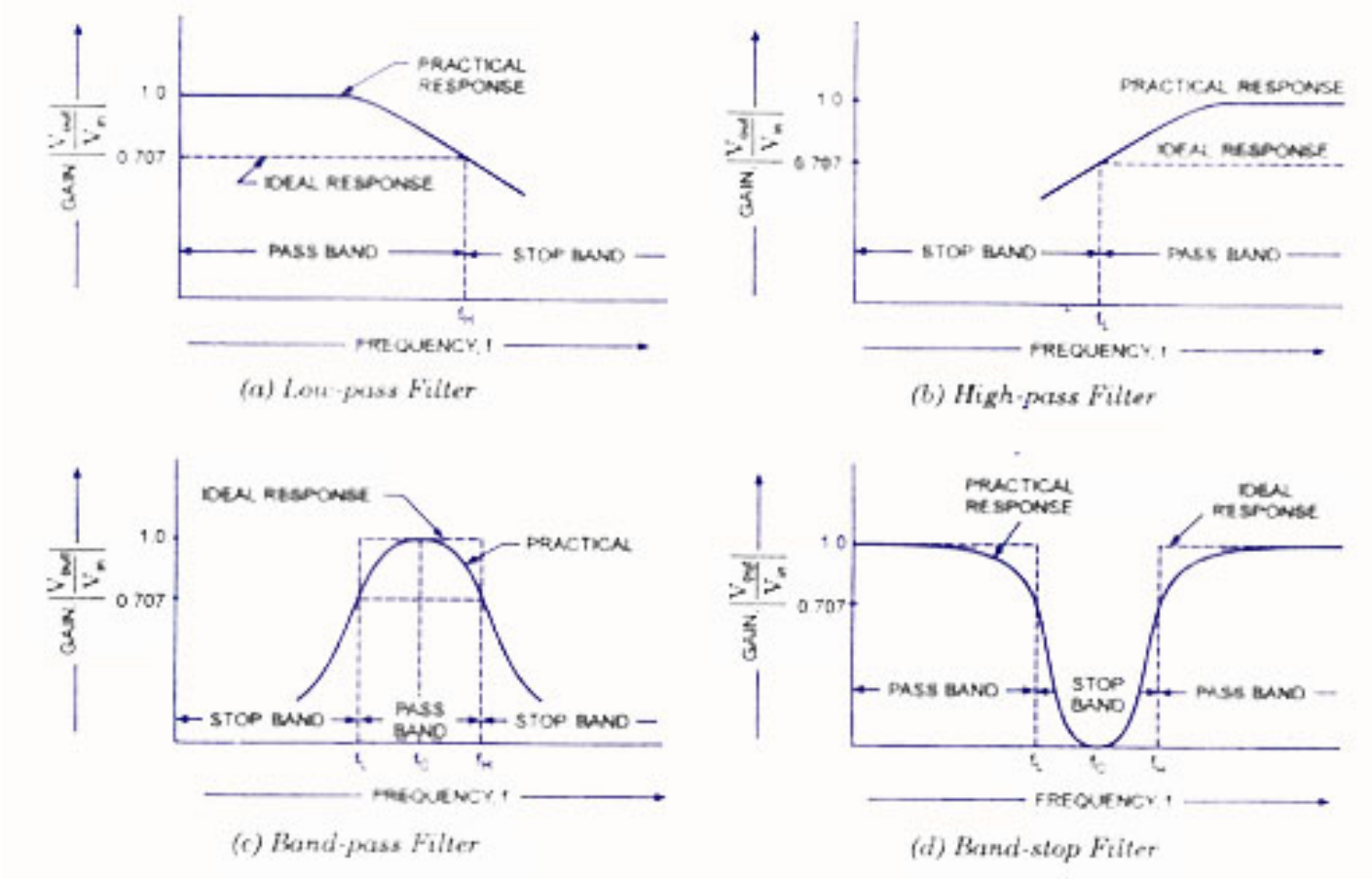


Figure 1: Magnitude response of a LPF,HPF,BPF,BSF,

Magnitude Characteristic of lowpass filter

The magnitude response can be expressed as

$$Magnitude = \begin{cases} 1-\delta_p \leq |H(j\Omega)| \leq 1 & for\ 0 \leq \Omega \leq \Omega_p \\ 0 \leq |H(j\Omega) \leq \delta_s & for\ |\Omega| \geq \Omega_s \end{cases}$$

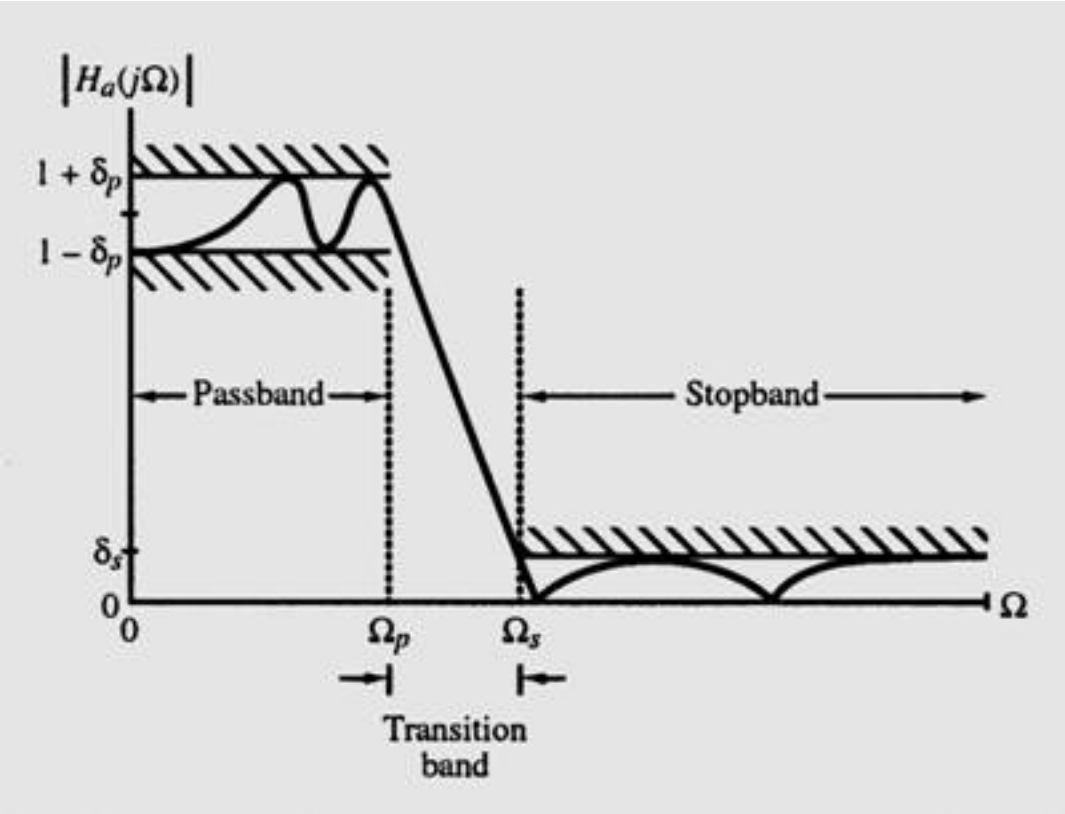


Figure 2: Magnitude response of a LPF

- $H(\Omega)$ cannot have an infinitely sharp cutoff from passband to stopband, that is $H(\Omega)$ cannot drop from unity to zero abruptly.
- It is not necessary to insist that the magnitude be constant in the entire passband of the filter. A small amount of ripple in the passband is usually tolerable.
- The filter response may not be zero in the stopband, it may have small nonzero value or ripple.
- The transition of the frequency response from passband to stopband defines transition band.
- The passband is usually called bandwidth of the filter.
- The width of transition band is $\Omega_s - \Omega_p$ where Ω_p defines passband edge frequency and Ω_s defines stopband edge frequency.
- The magnitude of passband ripple is varies between the limits $1 \pm \delta_p$ where δ_p is the ripple in the passband
- The ripple in the stopand of the filter is denoted as δ_p

Ω_p= Passband edge frequency in rad/second Ω_s= Stopband edge frequency in rad/second
ω_p= Passband edge frequency in rad/sample ω_s= Stopband edge frequency in rad/sample
A_p= Gain at passband edge frequency A_s= Gain at stopband edge frequency

$$\Omega_p = \frac{\omega_p}{T} \quad and \quad \Omega_s = \frac{\omega_s}{T}$$

where $T = \frac{1}{f_s}$= Sampling frequency

Butterworth Filter Design

The magnitude frequency response of Butterworth filter is

$$|H(j\Omega)|^2 = \frac{1}{\left[1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}\right]}$$

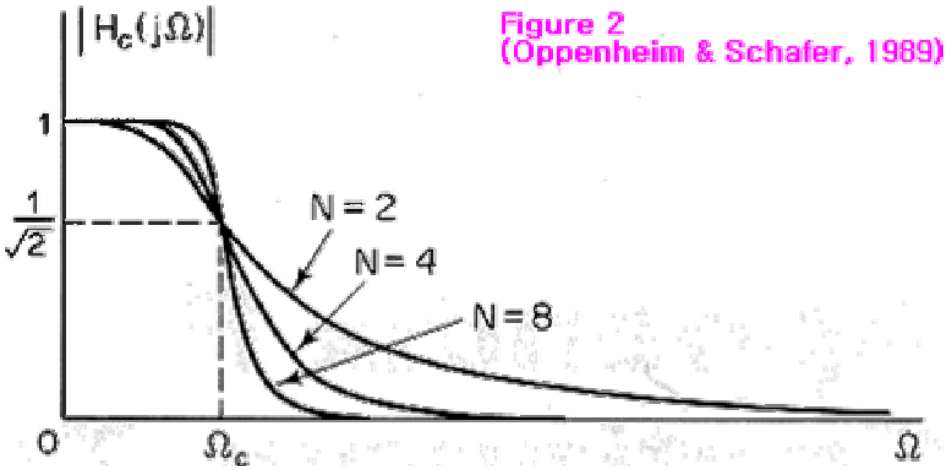


Figure 3: Frequency response of Butterworth low pass filter

Properties of butterworth filter

- $|H_N(j\Omega)|^2|_{\Omega=0} = 1$ for all N
- $|H_N(j\Omega)|^2|_{\Omega=\Omega_c} = 0.5$ for all finite N
- $|H_N(j\Omega)||_{\Omega=\Omega_c} = \frac{1}{\sqrt{2}} = 0.707$ $20log|H(j\Omega)||_{\Omega} = \Omega_c = -3.01$ dB
- $|H_N(j\Omega)|^2$ is a monotonically decreasing function of for Ω
- $|H_N(j\Omega)|^2$ approaches to ideal response as the value of N increases
- The filter is said to be normalized when cut-off frequency $\Omega_c = 1$ rad/sec.
- From normalized transfer function LPF, HPF, BPF BSF can be obtained by suitable transformation to the normalized LPF specification.

$|H_N(j\Omega)|^2 = H_N(j\Omega)H_N(-j\Omega) = \frac{1}{\left[1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}\right]}$

For normalized Butterworth lowpass filter $\Omega_c = 1$

$$H_N(j\Omega)H_N(-j\Omega) = \frac{1}{\left[1+(\Omega)^{2N}\right]}$$

Let $s = j\Omega \therefore \Omega = \frac{s}{j}$

$$H_N(s)H_N(-s) = \frac{1}{1+\left(\frac{s}{j}\right)^{2N}}$$

The poles of are determined by equating the denominator to zero

$$1+\left(\frac{s}{j}\right)^{2N} = 0$$

$$s = (-1)^{\frac{1}{2N}} j$$

-1 can be written as $e^{j\pi(2k+1)}$ where $k = 0, 1 \ldots$ and $j = e^{j\pi/2}$

$$s_k = e^{j\pi\frac{(2k+1)}{2N}} e^{j\pi/2} \qquad k = 0, 1 \ldots 2N-1$$

The poles are placed on a unit circle with radius unity and are placed at angles

$$\begin{aligned} s_k &= 1\angle\frac{k\pi}{N} \qquad k = 0, 1 \ldots 2N-1 \text{ when } N \text{ is odd} \\ &= 1\angle\frac{\pi}{2N} + \frac{k\pi}{N} \qquad k = 0, 1 \ldots 2N-1 \text{ when } N \text{ is even} \end{aligned}$$

N=1 ∴ k=0,1

$$S_k = 1\angle\frac{k\pi}{N}$$

$$S_0 = 1\angle 0, \quad S_1 = 1\angle\pi$$

The poles lying on left half of s plane is

$$H_N(s) = \frac{1}{\prod_{LHP}(s - s_k)} = \frac{1}{(s-(s_1))} = \frac{1}{(s+1)}$$

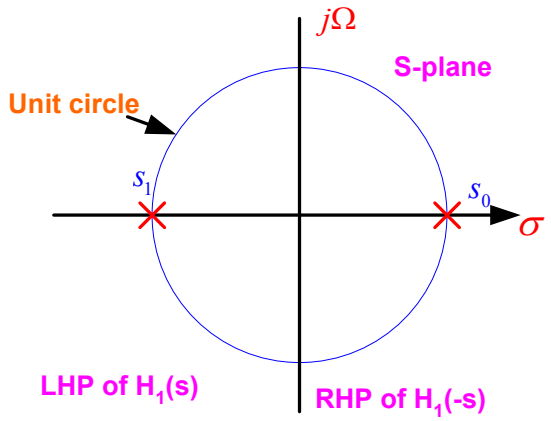


Figure 4: Poles of $H_1(s)H_1(-s)$

N=2 ∴ k=0,1,2,3 N is Even

$$S_k = 1\angle\frac{\pi}{2N} + \frac{k\pi}{N}$$

$$S_0 = 1\angle\frac{\pi}{4}, \quad S_1 = 1\angle\frac{3\pi}{4}$$

$$S_2 = 1\angle\frac{5\pi}{4}, \quad S_3 = 1\angle\frac{7\pi}{4}$$

The poles lying on left half of s plane is

$$\begin{aligned} H_N(s) &= \frac{1}{\prod_{LHP}[s - s_k]} = \frac{1}{[s-(s_1)][s-(s_2)]} \\ &= \frac{1}{[s-(-0.707+j0.707)][s-(-0.707-j0.707)]} \end{aligned}$$

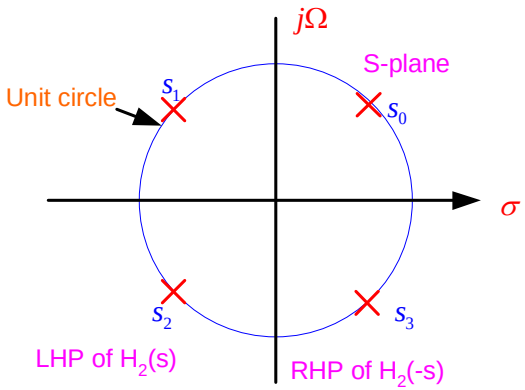


Figure 5: Poles of $H_2(s)H_2(-s)$

$$
\begin{aligned}
H_2(s) &= \frac{1}{[s+0.707-j0.707][s+0.707+j0.707]} \\
&= \frac{1}{[s+0.707]^2-[j0.707]^2} \\
&= \frac{1}{s^2+20.707s+(0.707)^2+(0.707)^2} \\
&= \frac{1}{s^2+1.414s+1}
\end{aligned}
$$

Determine the poles of lowpass Butterworth filter for N=3. Sketch the location of poles on s plane and hence determine the normalized transfer function of lowpass filter.

Solution:

N=3 $\therefore$ k=0,1,2,3,4,5
N is Odd

$$S_k = 1\angle\frac{k\pi}{N}$$

$$S_0 = 1\angle 0, \quad S_1 = 1\angle\frac{\pi}{3}, \quad S_2 = 1\angle\frac{2\pi}{3}$$

$$S_3 = 1\angle\pi, \quad S_4 = 1\angle\frac{4\pi}{3}, \quad S_5 = 1\angle\frac{5\pi}{3}$$

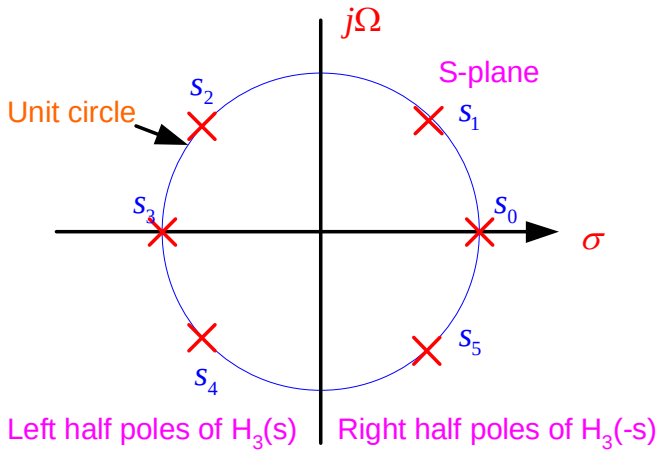


Figure 6: Poles of $H_3(s)H_3(-s)$

The poles lying on left half of s plane is

$$
\begin{aligned}
H_N(s) &= \frac{1}{\prod\limits_{LHP}[s-s_k]} = \frac{1}{[s-(s_2)][s-(s_3)][s-(s_4)]} \\
H_3(s) &= \frac{1}{[s-(-0.5+j0.866)][s-(-1)][s-(-0.5-j0.866)]} \\
&= \frac{1}{[s+1][s+0.5-j0.866][s+0.5+j0.866)]} \\
&= \frac{1}{[s+1][(s+0.5)^2-(j0.866)^2]]} = \frac{1}{(s+1)(s^2+s+1)}
\end{aligned}
$$

The poles are distributed on unit circle in the s plane
They are distributed half on the left half plane and half on the right half plane.

$$H_N(s) = \frac{1}{\prod_{LHP}(s - s_k)} = \frac{1}{B_N(s)}$$

Table 1: Normalized Butterworth Polynomial

Order N	Butterworth Polynomial
1	s+1
2	$s^2 + \sqrt{2}s + 1$
3	$(s^2 + s + 1)(s + 1)$
4	$(s^2 + 0.76536s + 1)(s^2 + 1.84776s + 1)$
5	$(s + 1)(s^2 + 0.6180s + 1)(s^2 + 1.6180s + 1)$

Design of Lowpass Butterworth Filter

The transfer function of normalized Butterworth lowpass filter is given by

$$H_N(s) = \frac{1}{\prod_{LHP}(s - s_k)} = \frac{1}{B_N(s)}$$

where $B_N(s)$ is nth order normalized Butterworth polynomial
The lowpass Butterworth filter has to meet the following frequency domain specifications

$$K_p \leq 20log|H(j\Omega)| \leq 0 \qquad for\ all\ \Omega \leq \Omega_p$$
$$20log|H(j\Omega)| \leq K_s \qquad for\ all\ \Omega \geq \Omega_s$$

K_p =Attenuation at passband frequency Ω_p in dB
K_s =Attenuation at stopband frequency Ω_s in dB

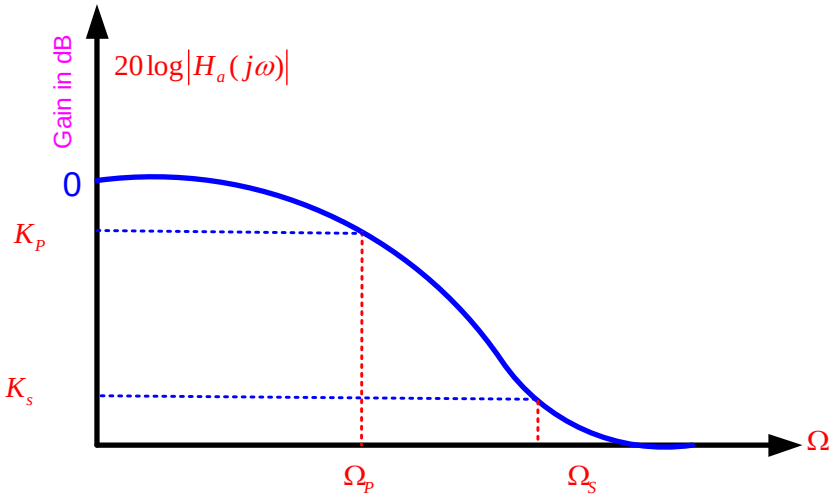


Figure 7: LPF specifications

The magnitude frequency response is

$$|H(j\Omega)| = \frac{1}{\left[1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}\right]^{\frac{1}{2}}}$$

Taking 20 log on both sides

$$\begin{aligned} 20\ log|H(j\Omega)| &= 20\ log\left[\frac{1}{\left[1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}\right]^{\frac{1}{2}}}\right] \\ &= -20\ log\left[1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}\right]^{\frac{1}{2}} \\ &= -10\ log\left[1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}\right] \end{aligned}$$

$\Omega = \Omega_p$ and $K = K_p$

$$K_p = -10\ log\left[1+\left(\frac{\Omega_P}{\Omega_c}\right)^{2N}\right]$$

$$\left[\frac{\Omega_P}{\Omega_c}\right]^{2N} = 10^{\frac{-K_p}{10}} - 1 \quad (1)$$

$\Omega = \Omega_s$ and $K = K_s$

$$K_S = -10\ log\left[1+\left(\frac{\Omega_S}{\Omega_c}\right)^{2N}\right]$$

$$\left[\frac{\Omega_S}{\Omega_c}\right]^{2N} = 10^{\frac{-K_S}{10}} - 1 \quad (2)$$

Dividing Equation 1 by Equation 2

$$\left[\frac{\Omega_p}{\Omega_S}\right]^{2N} = \frac{10^{\frac{-K_p}{10}} - 1}{10^{\frac{-K_S}{10}} - 1}$$

$$2N\ log\ \left[\frac{\Omega_p}{\Omega_S}\right] = log\ \left[\frac{10^{\frac{-K_p}{10}} - 1}{10^{\frac{-K_S}{10}} - 1}\right]$$

$$N = \frac{log\ \left[\frac{10^{\frac{-K_p}{10}} - 1}{10^{\frac{-K_S}{10}} - 1}\right]}{2log\ \left[\frac{\Omega_p}{\Omega_S}\right]}$$

where N is the order of the filter
The cutoff frequency Ω_C is

$$\Omega_C = \frac{\Omega_p}{\left(10^{\frac{-K_p}{10}} - 1\right)^{\frac{1}{2N}}}$$

OR

$$\Omega_C = \frac{\Omega_S}{\left(10^{\frac{-K_S}{10}} - 1\right)^{\frac{1}{2N}}}$$

Design steps for Butterworth Lowpass Filter

From the given specifications

1. Determine the order of the Filter using

$$N = \frac{log\left[\frac{10^{\frac{-K_p}{10}}-1}{10^{\frac{-K_S}{10}}-1}\right]}{2log\left[\frac{\Omega_p}{\Omega_S}\right]}$$

2. Determine the cutoff frequency Ω_C using

$$\Omega_C = \frac{\Omega_p}{\left(10^{\frac{-K_p}{10}}-1\right)^{\frac{1}{2N}}} \quad OR\ \Omega_C = \frac{\Omega_s}{\left(10^{\frac{-K_s}{10}}-1\right)^{\frac{1}{2N}}}$$

3. Determine the transfer function of normalized Butterworth filter by

$$H_N(s) = \frac{1}{\prod\limits_{LHP}(s-s_k)} = \frac{1}{B_N(s)}$$

4. From analog lowpass to lowpass frequency transformation, find the desired transfer function by substituting the following

$$H_a(s) = H_N(s)|_{s\rightarrow\frac{s}{\Omega_C}}$$

Design an analog Butterworth low pass filter to meet the following specifications T=1 second

$$\begin{aligned} 0.707 &= \leq |H(e^{j\omega})| \leq 1; \quad for\ 0 \leq \omega \leq 0.3\pi \\ &= |H(e^{j\omega})| \leq 0.2; \quad for\ .75\pi \leq \omega \leq \pi \end{aligned}$$

Solution:

Passband edge frequency $\omega_p = 0.3\pi\ rad/sample$
Stopband edge frequency $\omega_s = 0.75\pi\ rad/sample$
Passband edge analog frequency $\Omega_p = \frac{\omega_p}{1} = \frac{0.3\pi}{1} = 0.3\pi\ rad/second$
Stopband edge analog frequency $\Omega_s = \frac{\omega_s}{1} = \frac{0.75\pi}{1} = 0.75\pi\ rad/second$
K_p=20log(0.707)=-3.01 dB, K_s=20log(0.2)=-13.97 dB,
The order of the filter is

$$\begin{aligned} N &= \frac{log\left[\frac{10^{\frac{-K_p}{10}}-1}{10^{\frac{-K_S}{10}}-1}\right]}{2log\left[\frac{\Omega_p}{\Omega_S}\right]} \\ &= \frac{log\left[\frac{10^{\frac{3.01}{10}}-1}{10^{\frac{13.97}{10}}-1}\right]}{2log\left[\frac{0.3\pi}{0.75\pi}\right]} = \frac{log\left[\frac{1}{24}\right]}{2\times(-0.398)} \\ &= \frac{-1.38}{-0.796} = 1.7336 \simeq 2 \end{aligned}$$

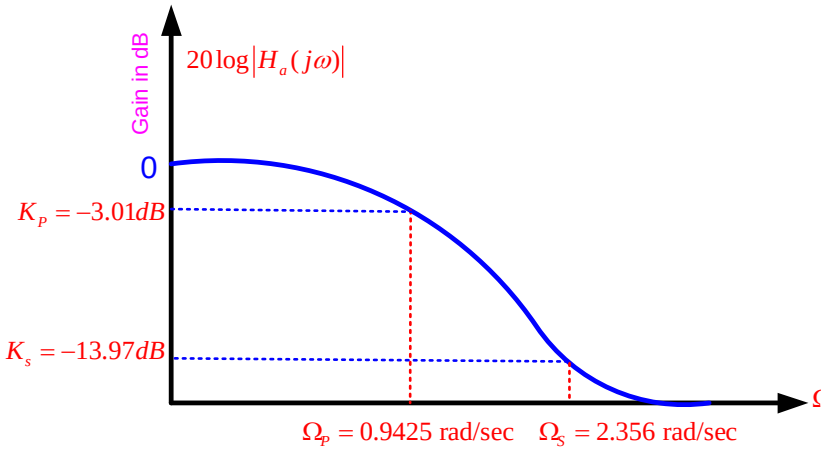


Figure 8: LPF specifications

For even N

$$H(s_n) = \prod_{k=1}^{\frac{N}{2}} \frac{1}{s^2 + b_k s + 1}$$

where $b_k = 2sin\left[\frac{(2k-1)\pi}{2N}\right]$

N=2

$k = \frac{N}{2} = \frac{2}{2} = 1$

k=1

$b_k = 2sin\left[\frac{(2-1)\pi}{2\times 2}\right]$=1.4142

$$H(s_n) = \frac{1}{s^2 + 1.4142s + 1}$$

$$\begin{aligned}\Omega_c &= \frac{\Omega_s}{(10^{\frac{-k_s}{10}} - 1)^{\frac{1}{2N}}} \\ &= \frac{2.3562}{(10^{\frac{13.97}{10}} - 1)^{\frac{1}{4}}} \\ &= 1.0664\ rad/sec\end{aligned}$$

OR

N=2 ∴ k=0,1,2,3 N is Even

$$S_k = 1\angle\frac{\pi}{2N} + \frac{k\pi}{N}$$

$$S_0 = 1\angle\frac{\pi}{4}, \quad S_1 = 1\angle\frac{3\pi}{4}$$

$$S_2 = 1\angle\frac{5\pi}{4}, \quad S_3 = 1\angle\frac{7\pi}{4}$$

The poles lying on left half of s plane

$$\begin{aligned}H_N(s) &= \frac{1}{\prod_{LHP}[s - s_k]} = \frac{1}{[s-(s_1)][s-(s_2)]} \\ &= \frac{1}{[s-(-0.707+j0.707)][s-(-0.707-j0.707)]} \\ &= \frac{1}{s^2 + 1.4142s + 1}\end{aligned}$$

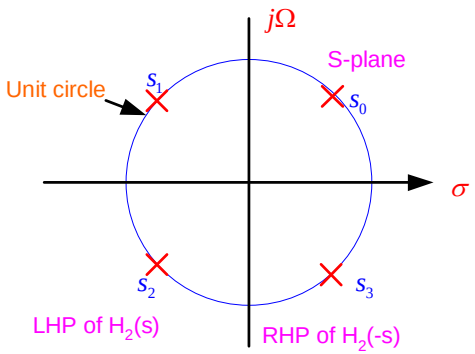


Figure 9: Poles of $H_2(s)H_2(-s)$

Unnomalized transfer function, H(s)

$$
\begin{aligned}
H_a(s) &= H_2(s)|_{s \to \frac{s}{\Omega_c}} \\
&= H_2(s)|_{s \to \frac{s}{1.0644}} \\
&= \frac{1}{\frac{s^2}{{\Omega_c}^2} + 1.4142\frac{s}{\Omega_c} + 1} \\
&= \frac{1}{\frac{s^2 + 1.4142\Omega_c s + {\Omega_c}^2}{\Omega_c^2}} \\
&= \frac{\Omega_c^2}{s^2 + 1.4142\Omega_c s + {\Omega_c}^2} \\
&= \frac{1.0644^2}{s^2 + 1.4142 \times 1.0644 s + 1.0644^2} \\
&= \frac{1.133}{s^2 + 1.5047 s + 1.133}
\end{aligned}
$$

Design an analog Butterworth low pass filter to meet the following specifications T=1 second

$$\begin{aligned} 0.9 &= \leq |H(e^{j\omega})| \leq 1; \quad for\ 0 \leq \omega \leq 0.35\pi \\ &= |H(e^{j\omega})| \leq 0.275; \quad for\ .7\pi \leq \omega \leq \pi \end{aligned}$$

Solution:

Passband edge frequency $\omega_p = 0.35\pi\ rad/sample$
Stopband edge frequency $\omega_s = 0.7\pi\ rad/sample$
Passband edge analog frequency $\Omega_p = \frac{\omega_p}{1} = \frac{0.35\pi}{1} = 0.35\pi\ rad/second$
Stopband edge analog frequency $\Omega_s = \frac{\omega_s}{1} = \frac{0.7\pi}{1} = 0.7\pi\ rad/second$
K_p=20log(0.9)=-0.9151 dB, K_s=20log(0.2)=-11.2133 dB,
The order of the filter is

$$\begin{aligned} N &= \frac{log\left[\frac{10^{\frac{-K_p}{10}}-1}{10^{\frac{-K_S}{10}}-1}\right]}{2log\left[\frac{\Omega_p}{\Omega_S}\right]} \\ &= \frac{log\left[\frac{10^{\frac{0.9151}{10}}-1}{10^{\frac{11.213}{10}}-1}\right]}{2log\left[\frac{0.35\pi}{0.7\pi}\right]} = \frac{log\left[\frac{0.234}{12.21}\right]}{2\times(-0.301)} \\ &= \frac{-1.717}{-0.602} = 2.852 \simeq 3 \end{aligned}$$

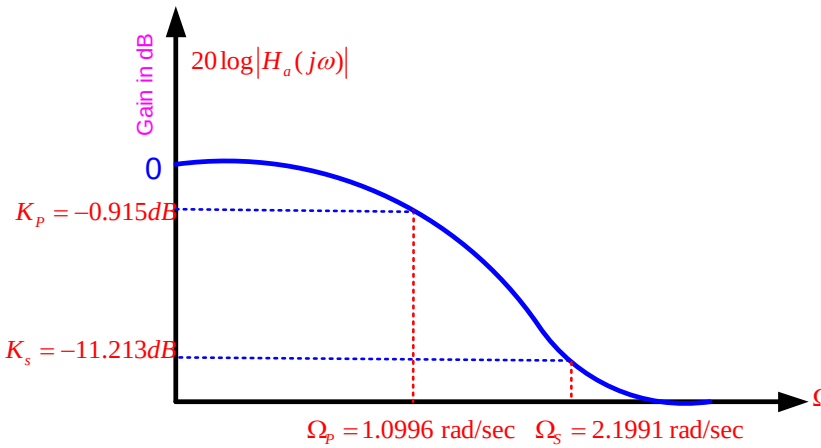


Figure 10: LPF specifications

For odd N=3

$$H(s_n) = \frac{1}{(s+1)} \prod_{k=1}^{\frac{N-1}{2}} \frac{1}{s^2 + b_k s + 1}$$

where $b_k = 2sin\left[\frac{(2k-1)\pi}{2N}\right]$

N=3

$k = \frac{N-1}{2} = \frac{3-1}{2} = 1$

k=1

$b_k = b_1 = 2sin\left[\frac{(2-1)\pi}{2\times 3}\right]$=1

$$\begin{aligned} H(s_n) &= \frac{1}{(s+1)(s^2+s+1)} \\ &= \frac{1}{s^3+2s^2+2s+1} \end{aligned}$$

$$\begin{aligned} \Omega_c &= \frac{\Omega_s}{(10^{\frac{-k_s}{10}} - 1)^{\frac{1}{2N}}} \\ &= \frac{2.2}{(10^{\frac{11.21}{10}} - 1)^{\frac{1}{6}}} = \frac{2.2}{1.515} \\ &= 1.45\ rad/sec \end{aligned}$$

OR

N=3 $\therefore$ k=0,1,2,3,4,5 N is Even

$$S_k = 1\angle \frac{\pi}{2N} + \frac{k\pi}{N}$$

$$S_0 = 1\angle \frac{\pi}{4}, \quad S_1 = 1\angle \frac{3\pi}{4}$$

$$S_2 = 1\angle \frac{5\pi}{4}, \quad S_3 = 1\angle \frac{7\pi}{4}$$

The poles lying on left half of s plane

$$\begin{aligned} H_N(s) &= \frac{1}{\prod_{LHP}[s - s_k]} = \frac{1}{[s-(s_1)][s-(s_2)]} \\ &= \frac{1}{[s-(-0.707+j0.707)][s-(-0.707-j0.707)]} \end{aligned}$$

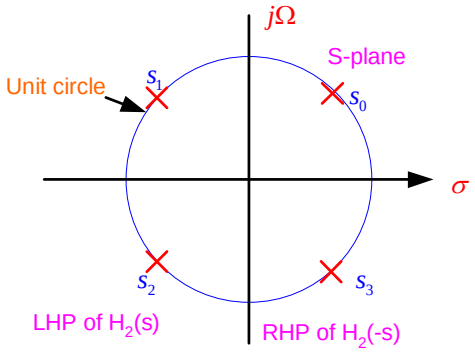


Figure 11: Poles of $H_2(s)H_2(-s)$

Unnomalized transfer function, H(s) and $\Omega_c = 1.45\ rad/sec$

$$
\begin{aligned}
H_a(s) &= H_3(s)|_{s\to\frac{s}{\Omega_c}} = \frac{1}{s^3+2s^2+2s+1}|_{s\to\frac{s}{\Omega_c}} \\
&= H_3(s)|_{s\to\frac{s}{\Omega_c}} \\
&= \frac{1}{\frac{s^3}{{\Omega_c}^3}+2\frac{s^2}{{\Omega_c}^2}+2\frac{s}{\Omega_c}+1} \\
&= \frac{1}{\frac{s^3+2\Omega_c s^2+2\Omega_c^2 s+\Omega_c^3}{\Omega_c^3}} \\
&= \frac{\Omega_c^3}{s^3+2\Omega_c s^2+2\Omega_c^2 s+\Omega_c^3} \\
&= \frac{1.45^3}{s^3+2\times 1.45 s^2+2\times 1.45^2 s+1.45^3} \\
&= \frac{3.048}{s^3+2.9s^2+4.205s+3.048}
\end{aligned}
$$

Design an analog Butterworth low pass filter to meet the following specifications T=1 second

$$\begin{aligned} 0.8 &= \leq |H(e^{j\omega})| \leq 1; \quad for\ 0 \leq \omega \leq 0.2\pi \\ &= |H(e^{j\omega})| \leq 0.2; \quad for\ .32\pi \leq \omega \leq \pi \end{aligned}$$

Solution:

Passband edge frequency $\omega_p = 0.2\pi\ rad/sample$
Stopband edge frequency $\omega_s = 0.32\pi\ rad/sample$
Passband edge analog frequency $\Omega_p = \frac{\omega_p}{1} = \frac{0.35\pi}{1} = 0.6283\ rad/second$
Stopband edge analog frequency $\Omega_s = \frac{\omega_s}{1} = \frac{0.7\pi}{1} = 1.0053\ rad/second$
K_p=20log(0.8)=-1.9 dB, K_s=20log(0.2)=-13.97 dB,
The order of the filter is

$$\begin{aligned} N &= \frac{log\left[\frac{10^{\frac{-K_p}{10}}-1}{10^{\frac{-K_S}{10}}-1}\right]}{2log\left[\frac{\Omega_p}{\Omega_S}\right]} \\ &= \frac{log\left[\frac{10^{\frac{1.9}{10}}-1}{10^{\frac{13.97}{10}}-1}\right]}{2log\left[\frac{0.6283}{1.0053}\right]} = \frac{log\left[\frac{0.548}{24}\right]}{2\times(-0.204)} \\ &= \frac{-1.641}{-0.408} = 4.023 \simeq 4 \end{aligned}$$

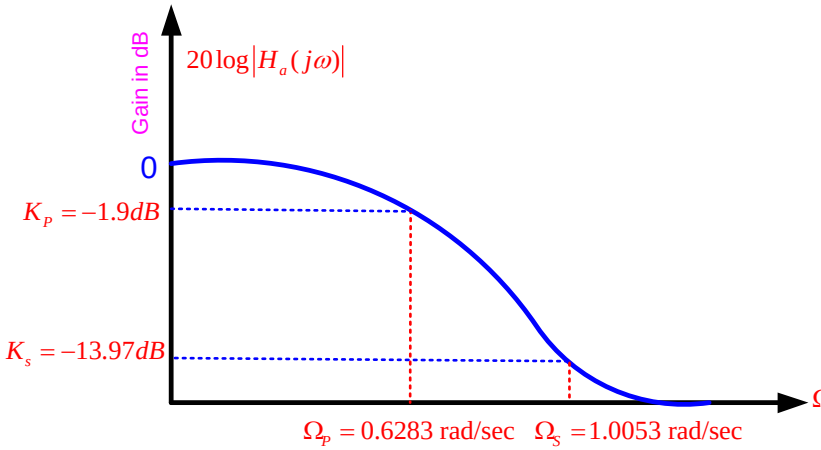


Figure 12: LPF specifications

For Even N=4

$$H(s_n) = \prod_{k=1}^{\frac{N}{2}} \frac{1}{s^2 + b_k s + 1}$$

where $b_k = 2sin\left[\frac{(2k-1)\pi}{2N}\right]$

N=4

$k = \frac{N}{2} = \frac{4}{2} = 2$

k=1

$b_k = b_1 = 2sin\left[\frac{(2-1)\pi}{2\times 4}\right]$=0.7654

k=2

$b_k = b_2 = 2sin\left[\frac{(4-1)\pi}{2\times 4}\right]$=1.8478

$$\begin{aligned} H(s_n) &= \frac{1}{(s^2 + 0.764s + 1)(s^2 + 1.8478s + 1)} \\ &= \frac{1}{s^4 + 2.6118s^3 + 3.4117s^2 + 2.6118s + 1} \end{aligned}$$

$$\begin{aligned} \Omega_c &= \frac{\Omega_s}{(10^{\frac{-k_s}{10}} - 1)^{\frac{1}{2N}}} \\ &= \frac{1.0053}{(10^{\frac{13.97}{10}} - 1)^{\frac{1}{8}}} = \frac{1.0053}{1.4873} \\ &= 0.676\ rad/sec \end{aligned}$$

Unnomalized transfer function, H(s) and $\Omega_c = 0.676\ rad/sec$

$$
\begin{aligned}
H_a(s) &= H_4(s)|_{s\to\frac{s}{\Omega_c}} = \frac{1}{s^4+2.6118s^3+3.4117s^2+2.6118s+1}|_{s\to\frac{s}{\Omega_c}} \\
&= H_4(s)|_{s\to\frac{s}{\Omega_c}} \\
&= \frac{1}{\frac{s^4}{{\Omega_c}^4}+2.6118\frac{s^3}{{\Omega_c}^3}+3.4117\frac{s^2}{{\Omega_c}^2}+2.6118\frac{s}{\Omega_c}+1} \\
&= \frac{1}{\frac{s^4+2.6118\Omega_c s^3+3.4117\Omega_c^2 s^2+2.6118\Omega_c^3 s+\Omega_c^4}{\Omega_c^4}} \\
&= \frac{\Omega_c^4}{s^4+2.6118\Omega_c s^3+3.4117\Omega_c^2 s^2+2.6118\Omega_c^3 s+\Omega_c^4} \\
&= \frac{0.676^4}{s^4+1.7655s^3+1.559s^2+0.8068s+0.2088} \\
&= \frac{0.2088}{s^4+1.7655s^3+1.559s^2+0.8068s+0.2088}
\end{aligned}
$$

Design an analog Butterworth low pass filter which has -2 dB attenuation at frequency 20 rad/sec and at least -10 dB attenuation at 30 rad/sec.

Solution:

Passband edge analog frequency $\Omega_p = 20\ rad/second$
Stopband edge analog frequency $\Omega_s = 30\ rad/second$
K_p=-2 dB, K_s=-10 dB,

The order of the filter is

$$N = \frac{log\left[\frac{10^{\frac{-K_p}{10}}-1}{10^{\frac{-K_S}{10}}-1}\right]}{2log\left[\frac{\Omega_p}{\Omega_S}\right]}$$

$$= \frac{log\left[\frac{10^{\frac{2}{10}}-1}{10^{\frac{10}{10}}-1}\right]}{2log\left[\frac{20}{30}\right]} = \frac{log\left[\frac{0.584}{9}\right]}{2\times(-0.176)}$$

$$= \frac{-1.1878}{-0.352} = 3.374 \simeq 4$$

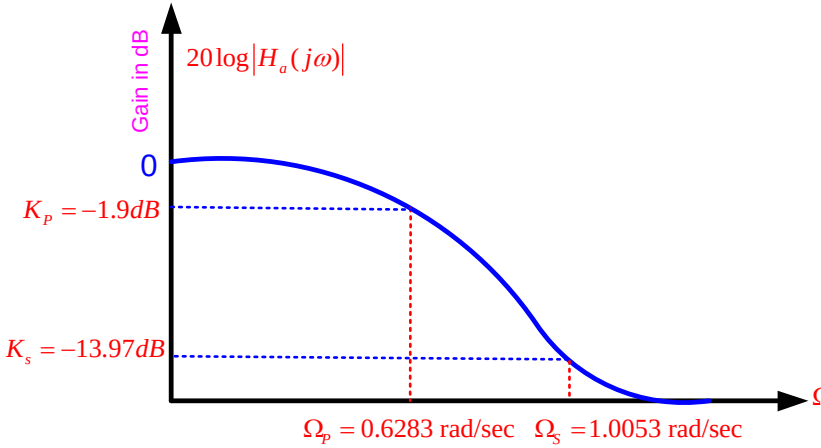


Figure 13: LPF specifications

For Even N=4

$$H(s_n) = \prod_{k=1}^{N/2} \frac{1}{s_n^2 + b_k s_n + 1}$$

where $b_k = 2sin\left[\frac{(2k-1)\pi}{2N}\right]$

N=4

$k = \frac{N}{2} = \frac{4}{2} = 2$

k=1

$b_k = b_1 = 2sin\left[\frac{(2-1)\pi}{2\times 4}\right]$=0.7654

k=2

$b_k = b_2 = 2sin\left[\frac{(4-1)\pi}{2\times 4}\right]$=1.8478

$$\begin{aligned} H(s_n) &= \frac{1}{(s^2 + 0.764s + 1)(s^2 + 1.8478s + 1)} \\ &= \frac{1}{s^4 + 2.6118s^3 + 3.4117s^2 + 2.6118s + 1} \end{aligned}$$

$$\begin{aligned} \Omega_c &= \frac{\Omega_s}{(10^{\frac{-k_s}{10}} - 1)^{\frac{1}{2N}}} \\ &= \frac{30}{(10^{\frac{10}{10}} - 1)^{\frac{1}{8}}} = \frac{30}{1.316} \\ &= 22.795\ rad/sec \end{aligned}$$

Unnomalized transfer function, H(s) and $\Omega_c = 22.795\ rad/sec$

$$
\begin{aligned}
H_a(s) &= H_4(s)|_{s\to\frac{s}{\Omega_c}} = \frac{1}{s^4+2.6118s^3+3.4117s^2+2.6118s+1}|_{s\to\frac{s}{22.795}} \\
&= \frac{1}{\frac{s^4}{{\Omega_c}^4}+2.6118\frac{s^3}{{\Omega_c}^3}+3.4117\frac{s^2}{{\Omega_c}^2}+2.6118\frac{s}{\Omega_c}+1} \\
&= \frac{1}{\frac{s^4+2.6118\Omega_c s^3+3.4117\Omega_c^2 s^2+2.6118\Omega_c^3 s+\Omega_c^4}{\Omega_c^4}} \\
&= \frac{\Omega_c^4}{s^4+2.6118\Omega_c s^3+3.4117\Omega_c^2 s^2+2.6118\Omega_c^3 s+\Omega_c^4} \\
&= \frac{22.795^4}{s^4+59.535s^3+1772.76s^2+30935.611s+22.795^4} \\
&= \frac{22.795^4}{s^4+59.535s^3+1772.76s^2+30935.611s+22.795^4}
\end{aligned}
$$

A Butterworth low pass filter has to meet the following specifications

i) passband gain K_P =1 dB at Ω_P =4 rad/sec

ii) Stop band attenuation greater than or equal to 20 dB at Ω_s =8 rad/sec

Determine the transfer function $H_a(s)$ of the lowest order Butterworth filter to meet the above the specifications

Solution:

Ω_p=4 rad/sec, Ω_s=8 rad/sec,
K_p=-1 dB, K_s=-20 dB,
The order of the filter is

$$N = \frac{log\left[\frac{10^{\frac{-K_p}{10}}-1}{10^{\frac{-K_S}{10}}-1}\right]}{2log\left[\frac{\Omega_p}{\Omega_S}\right]}$$

$$= \frac{log\left[\frac{10^{\frac{1}{10}}-1}{10^{\frac{20}{10}}-1}\right]}{2log\left[\frac{4}{8}\right]} = 4.289 \simeq 5$$

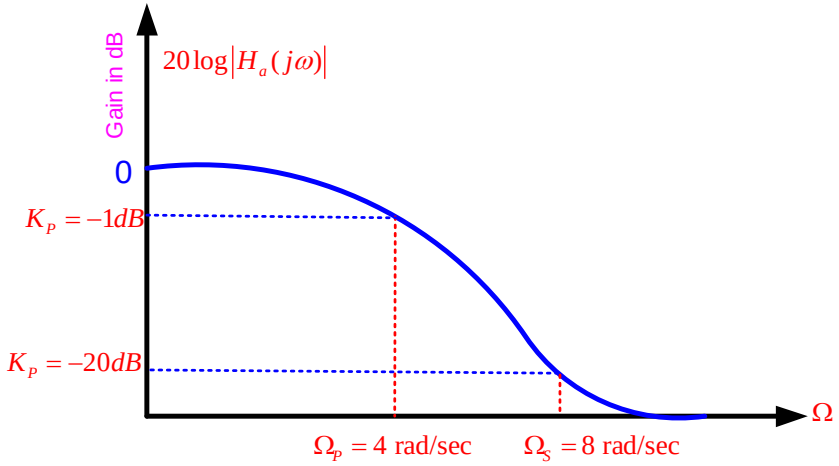


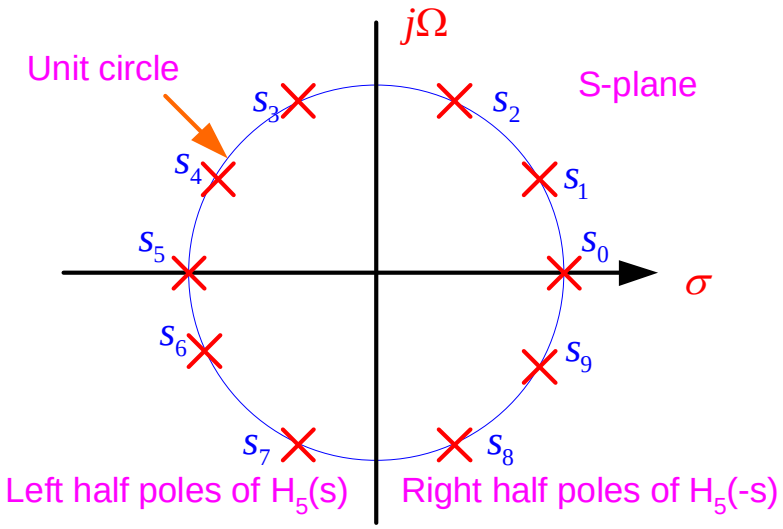
Figure 14: LPF specifications

$$S_k = 1\angle\theta_k \quad k = 0, 1 \ldots 2N-1$$

For odd N θ_k is

$$\theta_k = \frac{\pi k}{N}$$

$$\begin{aligned}
S_0 &= 1\underline{/0} = 1\underline{/0} = \\
S_1 &= 1\underline{/\frac{\pi}{5}} = 1\underline{/36^\circ} = 0.809 + j0.588 \\
S_2 &= 1\underline{/\frac{2\pi}{5}} = 1\underline{/72^\circ} = 0.309 + j0.951 \\
S_3 &= 1\underline{/\frac{3\pi}{5}} = 1\underline{/108^\circ} = -0.309 + j0.951 \\
S_4 &= 1\underline{/\frac{4\pi}{5}} = 1\underline{/144^\circ} = -0.809 - j0.588 \\
S_5 &= 1\underline{/\pi} = 1\underline{/180^\circ} = -1 \\
S_6 &= 1\underline{/\frac{6\pi}{5}} = 1\underline{/216^\circ} = -0.809 - j0.588 \\
S_7 &= 1\underline{/\frac{7\pi}{5}} = 1\underline{/252^\circ} = -0.309 - j0.951 \\
S_8 &= 1\underline{/\frac{8\pi}{5}} = 1\underline{/288^\circ} = 0.309 - j0.951 \\
S_9 &= 1\underline{/\frac{9\pi}{5}} = 1\underline{/324^\circ} = -0.809 - j0.588
\end{aligned}$$



$$
\begin{aligned}
H_5(s) &= \frac{1}{\prod\limits_{LHPonly} (s - s_k)} \\
&= \frac{1}{(s - s_3)(s - s_4)(s - s_5)(s - s_6)(s - s_7)} \\
&= \frac{1}{(s - 0.309 + j0.951)(s + 0.809 + j0.588)(s + 1)} \\
&\qquad (s + 0.809 - j0.588)(s + 0.309 + j0.951) \\
&= \frac{1}{[(s - 0.309)^2 + (0.951)^2][(s + 0.809)^2 + (0.588)^2](s + 1)} \\
&= \frac{1}{[(s^2 + 0.618s + 1)(s^2 + 1.618s + 1)(s + 1)} \\
&= \frac{1}{s^5 + 3.236s^4 + 5.236s^3 + 5.236s^2 + 3.236s + 1}
\end{aligned}
$$

$$\Omega_c = \frac{\Omega_p}{(10^{\frac{-k_p}{10}} - 1)^{\frac{1}{2N}}} = 4.5784 \ rad/sec$$

$$H_a(s) = H_5(s)|_{s \to \frac{s}{4.5787}}$$

$$
\begin{aligned}
H_a(s) &= H_5(s)|_{s \to \frac{s}{\Omega_c}} \\
&= H_5(s)|_{s \to \frac{s}{4.5787}} \\
&= \frac{1}{(\frac{s}{4.5787})^5 + 3.236(\frac{s}{4.5787})^4 + 5.236(\frac{s}{4.5787})^3 + 5.236(\frac{s}{4.5787})^2 + 3.236(\frac{s}{4.5787}) + 1} \\
&= \frac{2012.4}{s^5 + 14.82s^4 + 109.8s^3 + 502.6s^2 + 1422.36s + 2012.4}
\end{aligned}
$$

Verification

$$
\begin{aligned}
H_a(j\Omega) &= \frac{2012.4}{(j\Omega)^5 + 14.82(j\Omega)^4 + 109.8(j\Omega)^3 + 502.6(j\Omega)^2 + 1422.3(j\Omega) + 2012.4} \\
&= \frac{2012.4}{(14.82\Omega^4 - 502.6\Omega^2 + 2012.4) + j(\Omega^5 - 109.8\Omega^3 + 1422.3\Omega)} \\
|H_a(j\Omega)| &= \frac{2012.4}{\sqrt{(14.82\Omega^4 - 502.6\Omega^2 + 2012.4)^2 + j(\Omega^5 - 109.8\Omega^3 + 1422.3\Omega)^2}} \\
20\ log|H_a(j\Omega)|_4 &= -1\ dB \\
20\ log|H_a(j\Omega)|_8 &= -24\ dB
\end{aligned}
$$

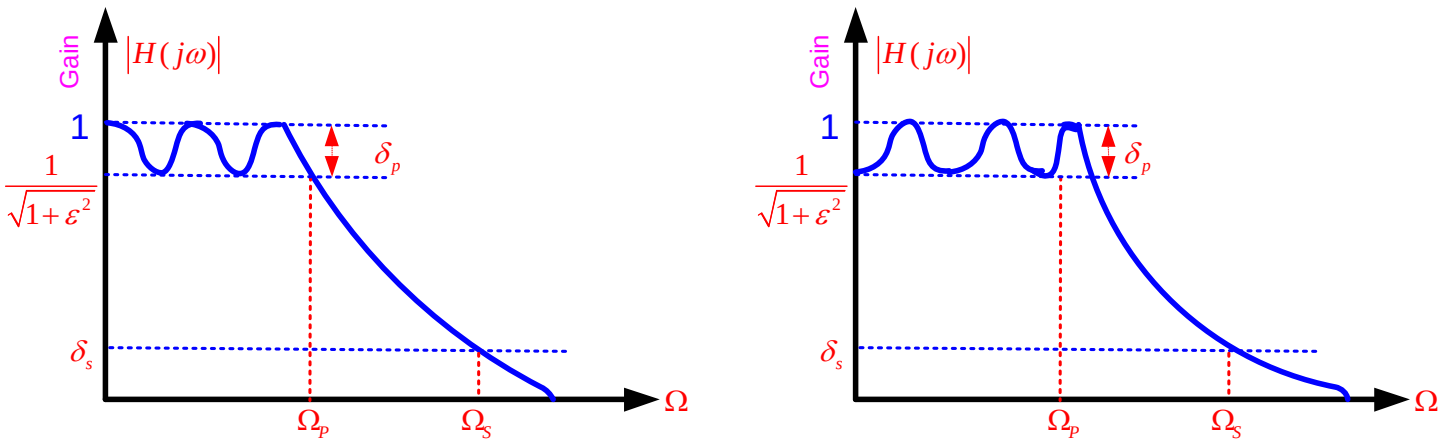
Chebyshev Filter Design

The magnitude frequency response of Chebyshev filter is

$$|H(j\Omega)|^2 = \frac{1}{\left[1 + \epsilon^2 T_n^2\left(\frac{\Omega}{\Omega_p}\right)\right]}$$

Properties of Chebyshev filter

- If $\Omega)_p = 1$ rad/sec then it is called as type-I normalized Chebyshev lowpass filter.
- $H_N(j\Omega)|^2|_{\Omega=0} = 1$ for all N
- $|H(j0)| = 1$ for odd N and $|H(j0)| = \frac{1}{\sqrt{1+\epsilon^2}}$ for even N
- The filter has uniform ripples in the passband and is monotonic outside the passband.
- The sum of the number of maxima and minima in the passband equals the order of the filter.



(a) N Odd (b) N Even

Figure 15: Magnitude frequency response of LPF for Chebyshev

Order of the Filter

K_p Gain or Magnitude at passband in normal value(without dB) for frequency Ω_p
K_s Gain or Magnitude at passband in normal value(without dB) for frequency Ω_s

$$N_1 = \frac{cosh^{-1}\left[\left[\frac{(1/K_s^2)-1}{(1/K_p^2)-1}\right]^{\frac{1}{2}}\right]}{cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)}$$

K_p Gain or Magnitude at passband in dB for frequency Ω_p
K_s Gain or Magnitude at passband in dB for frequency Ω_s

$$N_1 = \frac{cosh^{-1}\left[\left[\frac{10^{0.1K_s}-1}{10^{0.1K_p}-1}\right]^{\frac{1}{2}}\right]}{cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)}$$

Chose the order of the filter $N > N_1$

Normalized Chebyshev lowpass filter transfer function

When N is Even

$$H(s_n) = \prod_{k=1}^{\frac{N}{2}} \frac{B_k}{s^2 + b_k s + c_k}$$

When N is odd

$$H(s_n) = \frac{B_0}{s + c_0} \prod_{k=1}^{\frac{N-1}{2}} \frac{B_k}{s^2 + b_k s + c_k}$$

- where $b_k = 2y_N sin\left[\frac{(2k-1)\pi}{2N}\right]$, $c_k = y_N^2 + cos^2\left[\frac{(2k-1)\pi}{2N}\right]$
- $c_0 = y_N$

$$y_N = \frac{1}{2}\left[\left[\left(\frac{1}{\epsilon^2}+1\right)^{\frac{1}{2}} + \frac{1}{\epsilon}\right]^{\frac{1}{N}} - \left[\left(\frac{1}{\epsilon^2}+1\right)^{\frac{1}{2}} + \frac{1}{\epsilon}\right]^{-\frac{1}{N}}\right]$$

where $\epsilon = \left[(1/K_p^2) - 1\right]^{\frac{1}{2}}$

When N is Even the values of parameter B_k are evaluated using

$$H(s_n)|_{s=0} = \frac{1}{(1+\epsilon^2)^{\frac{1}{2}}}$$

When N is odd the values of parameter B_k are evaluated using

$$H(s_n)|_{s=0} = 1$$

Design steps for Chebyshev filter:

From the given specifications

1. Determine the order of the Filter
2. Determine the Normalized Chebyshev lowpass filter transfer function
3. From analog lowpass to lowpass frequency transformation, find the desired transfer function by substituting the following

$$H_a(s) = H_N(s)|_{s \to \frac{s}{\Omega_C}}$$

where $\Omega_C = \Omega_P$

Jan 2013, June 2015: Design a Chebyshev IIR analog low pass filter that has -3.0 dB frequency 100 rad/sec and stopband attenuation 25 dB or grater for all radian frequencies past 250 rad/sec

Solution:

Passband ripple K_p=-3.0 dB or in normal value is $K_p = 10^{K_p/20} = 10^{-3/20} = 0.707$
Stopband ripple K_s=25.0 dB or in normal value is $K_s = 10^{K_s/20} = 10^{-25/20} = 0.056$
Passband edge frequency =100 rad/sec Stopband edge frequency =250 rad/sec

$$
\begin{aligned}
N_1 &= \frac{cosh^{-1}\left[\left[\frac{(1/K_s^2)-1}{(1/K_p^2)-1}\right]^{\frac{1}{2}}\right]}{cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)} \\
&= \frac{cosh^{-1}\left[\left[\frac{(1/0.056^2)-1}{(1/0.707^2)-1}\right]^{\frac{1}{2}}\right]}{cosh^{-1}\left(\frac{250}{100}\right)} \\
&= \frac{cosh^{-1}\left[\frac{317}{1}\right]^{\frac{1}{2}}}{cosh^{-1}(2.5)} = \frac{cosh^{-1}[17.8]}{cosh^{-1}[2.5]} \\
&= \frac{3.57}{1.566} = 2.278 \simeq 3
\end{aligned}
$$

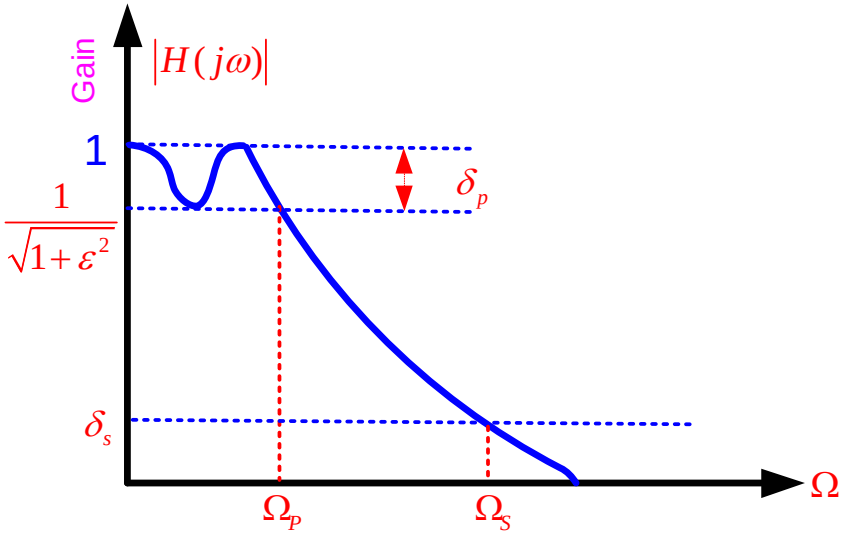


Figure 16: LPF specifications

N=3

When N is odd

$$H(s_n) = \frac{B_0}{s+c_0} \times \frac{B_k}{s^2+b_1 s+c_1}$$

$$H(s_n) = \frac{B_0}{s+c_0} \prod_{k=1}^{\frac{N-1}{2}} \frac{B_k}{s^2+b_k s+c_k}$$

$$\begin{aligned} \epsilon &= \left[(1/K_p^2)-1\right]^{\frac{1}{2}} \\ &= \left[(1/0.707^2)-1\right]^{\frac{1}{2}} = 1 \end{aligned}$$

N=3 $k = \frac{N-1}{2} = \frac{3-1}{2} = 1$

$$\begin{aligned} y_N &= \frac{1}{2}\left[\left[\left(\frac{1}{\epsilon^2}+1\right)^{\frac{1}{2}}+\frac{1}{\epsilon}\right]^{\frac{1}{N}} - \left[\left(\frac{1}{\epsilon^2}+1\right)^{\frac{1}{2}}+\frac{1}{\epsilon}\right]^{-\frac{1}{N}}\right] \\ &= \frac{1}{2}\left[\left[\left(\frac{1}{1^2}+1\right)^{\frac{1}{2}}+\frac{1}{1}\right]^{\frac{1}{3}} - \left[\left(\frac{1}{1^2}+1\right)^{\frac{1}{2}}+\frac{1}{1}\right]^{-\frac{1}{3}}\right] \\ &= \frac{1}{2}\left[\left[(2)^{\frac{1}{2}}+1\right]^{\frac{1}{3}} - \left[(2)^{\frac{1}{2}}+1\right]^{-\frac{1}{3}}\right] \\ &= \frac{1}{2}\left[[1.414+1]^{\frac{1}{3}} - [1.414+1]^{-\frac{1}{3}}\right] = \frac{1}{2}[1.341-0.745] \simeq 0.298 \end{aligned}$$

$C_0 = y_N = 0.298$ k=1 $b_k = 2 \times y_N sin\left[\frac{(2k-1)\pi}{2\times N}\right]$

k=1

$b_1 = 2 \times 0.298 sin\left[\frac{(2-1)\pi}{2\times 3}\right] = 0.298$

k=2

$$H(s_n) = \frac{B_0}{s+c_0} \times \frac{B_1}{s^2+b_1 s+c_1}$$

$c_k = y_N^2 + cos^2\left[\frac{(2k-1)\pi}{2N}\right]$

k=1

$$\begin{aligned} c_1 &= 0.298^2 + cos^2\left[\frac{(2-1)\pi}{2\times 3}\right] \\ &= 0.088 + cos^2\left[\frac{\pi}{6}\right] \\ &= 0.088 + \left[\frac{1+cos(\frac{2\pi}{6})}{2}\right] \\ &= 0.088 + 0.75 = 0.838 \end{aligned}$$

$$H(s_n) = \frac{B_0}{s+0.298} \times \frac{B_1}{s^2+0.298s+0.838}$$

When N is odd the values of parameter B_k are evaluated using

$$H(s_n)|_{s=0} = 1$$

$$H(s_n) = \frac{B_0 B_1}{0.298 \times 0.838} = 1$$

$$B_0 B_1 = 0.25$$

$B_0 = B_1$

$$B_0^2 = 0.25$$

$$B_0 = \sqrt{0.25} = 0.5$$

$$H(s_n) = \frac{B_0}{s + 0.298} \times \frac{B_1}{s^2 + 0.298s + 0.838}$$

$$H(s_n) = \frac{0.25}{s^3 + 0.596s^2 + 0.926s + 0.25}$$

Unnomalized transfer function, H(s) and $\Omega_p = 100\ rad/sec$

$$\begin{aligned}
H_a(s) &= H_3(s)|_{s \to \frac{s}{\Omega_p}} = \frac{0.25}{s^3 + 0.596s^2 + 0.926s + 0.25}|_{s \to \frac{s}{\Omega_p}} \\
&= H_3(s)|_{s \to \frac{s}{\Omega_p}} \\
&= \frac{0.25}{\frac{s^3}{{\Omega_p}^3} + 0.596\frac{s^2}{{\Omega_p}^2} + 0.926\frac{s}{\Omega_p} + 0.25} \\
&= \frac{0.25}{\frac{s^3 + 0.596\Omega_p s^2 + 0.926\Omega_c^2 s + 0.25\Omega_p^3}{\Omega_p^3}} \\
&= \frac{0.25 \times \Omega_p^3}{s^3 + 0.596\Omega_p s^2 + 0.926\Omega_c^2 s + 0.25\Omega_p^3} \\
&= \frac{0.25 \times 100^3}{s^3 + 0.596 \times 100 s^2 + 0.926 \times 100^2 s + 0.25 \times 100^3} \\
&= \frac{0.25 \times 100^3}{s^3 + 59.6s^2 + 926s + 0.25 \times 100^3}
\end{aligned}$$

Design a Chebyshev IIR low pass filter that has to meet the following specifications

i) passband ripple $\leq$0.9151 dB and passband edge frequency 0.25π rad/sec
ii) Stopband attenuation $\geq$12.395 dB and Stopband edge frequency 0.5π rad/sec

Solution:

Passband ripple K_p=0.9151 dB
or in normal value is $K_p = 10^{K_p/20} = 10^{-9151/20} = 0.9$
Stopband ripple K_s=12.395 dB
or in normal value is $K_s = 10^{K_s/20} = 10^{-12.395/20} = 0.24$
Passband edge frequency 0.25π=0.7854 rad/sec
Stopband edge frequency 0.5π =1.5708rad/sec

$$N_1 = \frac{cosh^{-1}\left[\left[\frac{(1/K_s^2)-1}{(1/K_p^2)-1}\right]^{\frac{1}{2}}\right]}{cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$= \frac{cosh^{-1}\left[\left[\frac{(1/0.24^2)-1}{(1/0.9^2)-1}\right]^{\frac{1}{2}}\right]}{cosh^{-1}\left(\frac{1.5708}{0.7854}\right)}$$

$$= \frac{cosh^{-1}\left[\frac{16.3611}{0.2346}\right]^{\frac{1}{2}}}{cosh^{-1}\left(\frac{1.5708}{0.7854}\right)} = \frac{cosh^{-1}[8.35]}{cosh^{-1}[2]} = \frac{2.8118}{1.3169} = 2.135 \simeq 3$$

N=3

When N is odd

$$H(s_n) = \frac{B_0}{s+c_0} \times \frac{B_k}{s^2+b_k s+c_k}$$

$$H(s_n) = \frac{B_0}{s+c_0} \prod_{k=1}^{\frac{N-1}{2}} \frac{B_k}{s^2+b_k s+c_k}$$

$$\begin{aligned} \epsilon &= \left[(1/K_p^2)-1\right]^{\frac{1}{2}} \\ &= \left[(1/0.9^2)-1\right]^{\frac{1}{2}} = 0.4843 \end{aligned}$$

N=3 $k = \frac{N-1}{2} = \frac{3-1}{2} = 1$

$$\begin{aligned} y_N &= \frac{1}{2}\left[\left[\left(\frac{1}{\epsilon^2}+1\right)^{\frac{1}{2}}+\frac{1}{\epsilon}\right]^{\frac{1}{N}} - \left[\left(\frac{1}{\epsilon^2}+1\right)^{\frac{1}{2}}+\frac{1}{\epsilon}\right]^{-\frac{1}{N}}\right] \\ &= \frac{1}{2}\left[\left[\left(\frac{1}{0.4843^2}+1\right)^{\frac{1}{2}}+\frac{1}{0.4843}\right]^{\frac{1}{3}} - \left[\left(\frac{1}{0.4843^2}+1\right)^{\frac{1}{2}}+\frac{1}{0.4843}\right]^{-\frac{1}{3}}\right] \\ &= \frac{1}{2}\left[\left[(5.2635)^{\frac{1}{2}}+2.064\right]^{\frac{1}{3}} - \left[(5.2635)^{\frac{1}{2}}+2.064\right]^{-\frac{1}{3}}\right] \\ &= \frac{1}{2}\left[[2.294+2.064]^{\frac{1}{3}} - [2.294+2.064]^{-\frac{1}{3}}\right] = \frac{1}{2}[1.6334-0.6122] \simeq 0.5107 \end{aligned}$$

$C_0 = y_N = 0.5107$ k=1 $b_k = 2 \times 0.5107 sin\left[\frac{(2-1)\pi}{2\times 3}\right] = 0.5107$

$c_k = y_N^2 + cos^2\left[\frac{(2k-1)\pi}{2N}\right]$

k=1

$$\begin{aligned} c_k &= 0.5107^2 + cos^2\left[\frac{(2-1)\pi}{2\times 3}\right] \\ &= 0.5107^2 + cos^2\left[\frac{\pi}{6}\right] \\ &= 0.5107^2 + \left[\frac{1+cos(\frac{2\pi}{6})}{2}\right] \\ &= 0.2608 + 0.75 = 1.0108 \end{aligned}$$

$$H(s_n) = \frac{B_0}{s+c_0} \times \frac{B_1}{s^2+b_k s+c_k}$$

$$H(s_n) = \frac{B_0}{s+0.5107} \times \frac{B_1}{s^2+b_1 s+c_1}$$

$$H(s_n) = \frac{B_0}{s+0.5107} \times \frac{B_1}{s^2+0.5107s+1.0108}$$

When N is odd the values of parameter B_k are evaluated using

$$H(s_n)|_{s=0} = 1$$

$$H(s_n) = \frac{B_0 B_1}{0.5107 \times 1.0108} = 1.9372 B_0 B_1 = 1$$

$$B_0 B_1 = \frac{1}{1.9372} = 0.5162$$

$B_0 = B_1$

$$B_0^2 = 0.5162$$

$$B_0 = \sqrt{0.5162} = 0.7185$$

$$H(s_n) = \frac{B_0}{s+0.5107} \times \frac{B_1}{s^2+0.5107s+1.0108} = \frac{0.7185}{s+0.5107} \times \frac{0.7185}{s^2+0.5107s+1.0108}$$

$$H(s_n) = \frac{0.5162}{s^3+1.0214s^2+1.2716s+0.5162}$$

Unnomalized transfer function, H(s) and $\Omega_p = 0.7854\ rad/sec$

$$\begin{aligned}
H_a(s) &= H_3(s)|_{s\to\frac{s}{\Omega_p}} = \frac{0.5162}{s^3+1.0214s^2+1.2716s+0.5162}|_{s\to\frac{s}{\Omega_p}} \\
&= \frac{0.5162}{\frac{s^3}{{\Omega_p}^3}+1.0214\frac{s^2}{{\Omega_p}^2}+1.2716\frac{s}{\Omega_p}+0.5162} \\
&= \frac{0.5162}{\frac{s^3+1.0214\Omega_p s^2+1.2716\Omega_p^2 s+\Omega_p^3}{\Omega_p^3}} \\
&= \frac{\Omega_p^3}{s^3+1.0214\Omega_p s^2+1.2716\Omega_p^2 s+\Omega_p^3} \\
&= \frac{0.5162\times 0.7854^3}{s^3+1.0214\times 0.7854s^2+1.2716\times 0.7854^2 s+0.7854^3} \\
&= \frac{0.250}{s^3+0.80229s^2+0.7844s+0.2501}
\end{aligned}$$

Design a Chebyshev IIR low pass filter that has to meet the following specifications

i) passband ripple $\leq$1.0 dB and passband edge frequency 1 rad/sec

ii) Stopband attenuation $\geq$15.0 dB and Stopband edge frequency 1.5 rad/sec

Solution:

Passband ripple K_p=1.0 dB or in normal value is $K_p = 10^{K_p/20} = 10^{-1/20} = 0.891$
Stopband ripple K_s=15.0 dB or in normal value is $K_s = 10^{K_s/20} = 10^{-15/20} = 0.177$
Passband edge frequency =1 rad/sec Stopband edge frequency =1.5rad/sec

$$
\begin{aligned}
N_1 &= \frac{cosh^{-1}\left[\left[\frac{(1/K_s^2)-1}{(1/K_p^2)-1}\right]^{\frac{1}{2}}\right]}{cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)} \\
&= \frac{cosh^{-1}\left[\left[\frac{(1/0.177^2)-1}{(1/0.891^2)-1}\right]^{\frac{1}{2}}\right]}{cosh^{-1}\left(\frac{1.5}{1.0}\right)} \\
&= \frac{cosh^{-1}\left[\frac{31.0}{0.26}\right]^{\frac{1}{2}}}{cosh^{-1}(1.5)} = \frac{cosh^{-1}[11.0]}{cosh^{-1}[1.5]} \\
&= \frac{3.08}{0.96} = 3.2 \simeq 4
\end{aligned}
$$

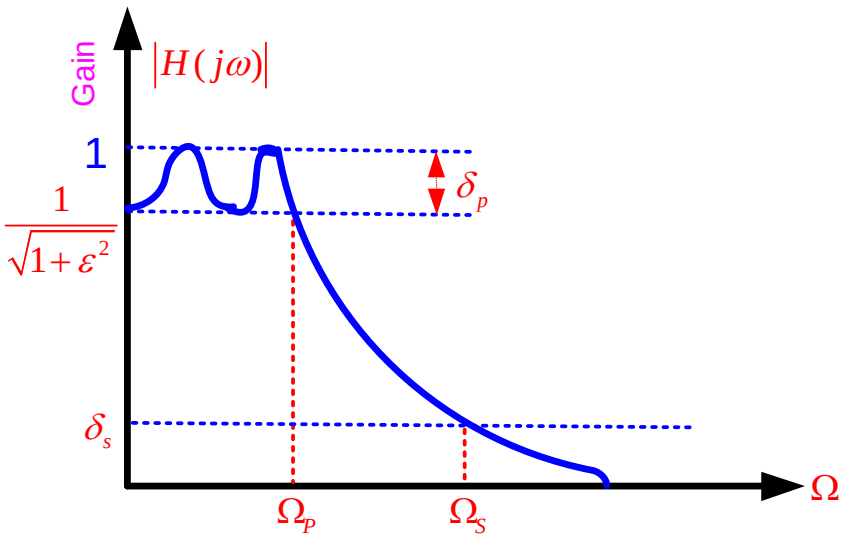


Figure 17: LPF specifications

N=4

When N is Even

$$H(s_n) = \frac{B_1}{s^2 + b_1 s + c_1} \times \frac{B_2}{s^2 + b_2 s + c_2}$$

$$H(s_n) = \prod_{k=1}^{\frac{N}{2}} \frac{B_k}{s^2 + b_k s + c_k}$$

N=4 $k = \frac{N}{2} = \frac{4}{2} = 2$

$$\begin{aligned} \epsilon &= \left[(1/K_p^2) - 1\right]^{\frac{1}{2}} \\ &= \left[(1/0.891^2) - 1\right]^{\frac{1}{2}} = 0.51 \end{aligned}$$

$$\begin{aligned} y_N &= \frac{1}{2}\left[\left[\left(\frac{1}{\epsilon^2}+1\right)^{\frac{1}{2}} + \frac{1}{\epsilon}\right]^{\frac{1}{N}} - \left[\left(\frac{1}{\epsilon^2}+1\right)^{\frac{1}{2}} + \frac{1}{\epsilon}\right]^{-\frac{1}{N}}\right] \\ &= \frac{1}{2}\left[\left[\left(\frac{1}{0.51^2}+1\right)^{\frac{1}{2}} + \frac{1}{0.51}\right]^{\frac{1}{4}} - \left[\left(\frac{1}{0.51^2}+1\right)^{\frac{1}{2}} + \frac{1}{0.51}\right]^{-\frac{1}{4}}\right] \\ &= \frac{1}{2}\left[\left[(4.84)^{\frac{1}{2}} + 1.96\right]^{\frac{1}{4}} - \left[(4.84)^{\frac{1}{2}} + 1.96\right]^{-\frac{1}{4}}\right] \\ &= \frac{1}{2}\left[[2.2 + 1.96]^{\frac{1}{4}} - [2.2 + 1.96]^{-\frac{1}{4}}\right] = \frac{1}{2}[1.428 - 0.7] \simeq 0.364 \end{aligned}$$

$C_0 = y_N = 0.364$ k=1 $b_k = 2 \times y_N sin\left[\frac{(2k-1)\pi}{2\times N}\right]$

k=1

$b_1 = 2 \times 0.364 sin\left[\frac{(2-1)\pi}{2\times 4}\right] = 0.278$

k=2

$b_2 = 2 \times 0.364 sin\left[\frac{(2\times 2-1)\pi}{2\times 4}\right] = 0.672$

$c_k = y_N^2 + cos^2\left[\frac{(2k-1)\pi}{2N}\right]$

k=1

$$\begin{aligned} c_1 &= 0.364^2 + cos^2\left[\frac{(2-1)\pi}{2\times 4}\right] \\ &= 0.132 + cos^2\left[\frac{\pi}{8}\right] \\ &= 0.132 + \left[\frac{1 + cos(\frac{2\pi}{8})}{2}\right] \\ &= 0.132 + 0.853 \\ &= 0.132 + 0.853 = 0.985 \end{aligned}$$

k=2

$$\begin{aligned} c_2 &= 0.364^2 + cos^2\left[\frac{(2\times 2-1)\pi}{2\times 4}\right] \\ &= 0.132 + cos^2\left[\frac{3\pi}{8}\right] \\ &= 0.132 + \left[\frac{1 + cos(\frac{6\pi}{8})}{2}\right] \\ &= 0.132 + 0.146 = 0.278 \end{aligned}$$

$$H(s_n) = \frac{B_1}{s^2 + b_1 s + c_1} \times \frac{B_2}{s^2 + b_2 s + c_2}$$

$$H(s_n) = \frac{B_1}{s^2 + 0.278s + 0.985} \times \frac{B_2}{s^2 + 0.672s + 0.278}$$

When N is odd the values of parameter B_k are evaluated using

$$H(s_n)|_{s=0} = \frac{1}{(1+\epsilon^2)^{1/2}} = \frac{1}{(1+0.51^2)^{1/2}} = 0.89$$

$$H(s_n) = \frac{B_1 B_2}{0.985 \times 0.278} = 0.89$$

$$B_1 B_2 = 0.244$$

$B_1 = B_2$

$$B_1^2 = 0.244$$

$$B_1 = \sqrt{0.264} = 0.493$$

$$H(s_n) = \frac{B_1}{s^2 + 0.278s + 0.985} \times \frac{B_2}{s^2 + 0.672s + 0.278} = \frac{0.493}{s^2 + 0.278s + 0.985} \times \frac{0.493}{s^2 + 0.672s + 0.278}$$

$$H(s_n) = \frac{0.243}{s^4 + 0.95s^3 + 1.45s^2 + 1.434s + 0.2738}$$

Unnomalized transfer function, H(s) and $\Omega_p = 1.0\ rad/sec$

$$\begin{aligned} H_a(s) &= H_4(s)|_{s \to \frac{s}{\Omega_p}} = \frac{0.243}{s^4 + 0.95s^3 + 1.45s^2 + 1.434s + 0.2738}|_{s \to \frac{s}{1}} \\ &= \frac{0.263}{s^4 + 0.95s^3 + 1.45s^2 + 1.434s + 0.2738} \end{aligned}$$

July 2014, Dec 2014 Design a Chebyshev IIR low pass filter that has to meet the following specifications

i) passband ripple $\leq$2 dB and passband edge frequency 1 rad/sec
ii) Stopband attenuation $\geq$20 dB and Stopband edge frequency 1.3 rad/sec

Solution:

Passband ripple K_p=2 dB
or in normal value is $K_p = 10^{K_p/20} = 10^{-2/20} = 0.7943$
Stopband ripple K_s=12.395 dB
or in normal value is $K_s = 10^{K_s/20} = 10^{-20/20} = 0.1$
Passband edge frequency 1 rad/sec
Stopband edge frequency 1.3 rad/sec

$$N_1 = \frac{cosh^{-1}\left[\left[\frac{(1/K_s^2)-1}{(1/K_p^2)-1}\right]^{\frac{1}{2}}\right]}{cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$= \frac{cosh^{-1}\left[\left[\frac{(1/0.1^2)-1}{(1/0.7943^2)-1}\right]^{\frac{1}{2}}\right]}{cosh^{-1}\left(\frac{1.3}{1.0}\right)}$$

$$= \frac{cosh^{-1}\left[\frac{99.0}{0.585}\right]^{\frac{1}{2}}}{cosh^{-1}\left(\frac{1.3}{1.0}\right)} = \frac{cosh^{-1}[13.00]}{cosh^{-1}[1.3]} = \frac{3.256}{0.756} = 4.3 \simeq 5$$

When N is odd

$$H(s_n) = \frac{B_0}{s+c_0} \times \frac{B_1}{s^2+b_1 s+c_1} \times \frac{B_2}{s^2+b_2 s+c_2}$$

$$H(s_n) = \frac{B_0}{s+c_0} \prod_{k=1}^{\frac{N-1}{2}} \frac{B_k}{s^2+b_k s+c_k}$$

$$\begin{aligned} \epsilon &= \left[(1/K_p^2)-1\right]^{\frac{1}{2}} \\ &= \left[(1/0.7943^2)-1\right]^{\frac{1}{2}} = 0.7648 \end{aligned}$$

N=5 $k = \frac{N-1}{2} = \frac{5-1}{2} = 2$

$$\begin{aligned} y_N &= \frac{1}{2}\left[\left[\left(\frac{1}{\epsilon^2}+1\right)^{\frac{1}{2}} + \frac{1}{\epsilon}\right]^{\frac{1}{N}} - \left[\left(\frac{1}{\epsilon^2}+1\right)^{\frac{1}{2}} + \frac{1}{\epsilon}\right]^{-\frac{1}{N}}\right] \\ &= \frac{1}{2}\left[\left[\left(\frac{1}{0.7648^2}+1\right)^{\frac{1}{2}} + \frac{1}{0.7648}\right]^{\frac{1}{5}} - \left[\left(\frac{1}{0.7648^2}+1\right)^{\frac{1}{2}} + \frac{1}{0.7648}\right]^{-\frac{1}{5}}\right] \\ &= \frac{1}{2}\left[\left[(2.71)^{\frac{1}{2}} + 1.307\right]^{\frac{1}{5}} - \left[(2.71)^{\frac{1}{2}} + 1.307\right]^{-\frac{1}{5}}\right] \\ &= \frac{1}{2}\left[[1.646+1.307]^{\frac{1}{5}} - [1.646+1.307]^{-\frac{1}{5}}\right] = \frac{1}{2}[1.241-0.805] \simeq 0.218 \end{aligned}$$

$C_0 = y_N = 0.218$ $\quad b_k = 2 \times y_N sin\left[\frac{(2k-1)\pi}{2\times N}\right]$ k=1 $\quad b_1 = 2 \times 0.218 sin\left[\frac{(2-1)\pi}{2\times 5}\right] = 0.134$

k=2

$b_2 = 2 \times 0.218 sin\left[\frac{(4-1)\pi}{2\times 5}\right] = 0.352$

$c_k = y_N^2 + cos^2\left[\frac{(2k-1)\pi}{2N}\right]$

k=1

$$\begin{aligned} c_1 &= 0.218^2 + cos^2\left[\frac{(2-1)\pi}{2\times 5}\right] \\ &= 0.047 + cos^2\left[\frac{\pi}{10}\right] \\ &= 0.047 + \left[\frac{1+cos(\frac{2\pi}{10})}{2}\right] \\ &= 0.047 + 0.904 = 0.951 \end{aligned}$$

k=2

$$\begin{aligned} c_2 &= 0.218^2 + cos^2\left[\frac{(4-1)\pi}{2\times 5}\right] \\ &= 0.047 + cos^2\left[\frac{3\pi}{10}\right] \\ &= 0.047 + \left[\frac{1+cos(\frac{2\times 3\pi}{10})}{2}\right] \\ &= 0.047 + 0.345 = 0.392 \end{aligned}$$

$$H(s_n) = \frac{B_0}{s+c_0} \times \frac{B_1}{s^2+b_1 s+c_1} \times \frac{B_2}{s^2+b_2 s+c_2}$$

$$H(s_n) = \frac{B_0}{s+0.218} \times \frac{B_1}{s^2+0.134s+0.951} \times \frac{B_2}{s^2+0.352s+0.392}$$

When N is odd the values of parameter B_k are evaluated using

$$H(s_n)|_{s=0} = 1$$

When N is odd the values of parameter B_k are evaluated using

$$H(s_n)|_{s=0} = 1$$

$$H(s_n) = \frac{B_0 B_1 B_2}{0.218 \times 0.951 \times 0.392} = 12.3 B_0 B_1 B_2 = 1$$

$$B_0 B_1 B_2 = \frac{1}{12.3} = 0.081$$

$B_0 = B_1 = B_2$ Then $B_0^3 = 0.081$ $B_0 = \sqrt[3]{0.081} = 0.081^{\frac{1}{3}} = 0.432$

$$H(s_n) = \frac{0.432}{s + 0.218} \times \frac{0.432}{s^2 + 0.134s + 0.951} \times \frac{0.432}{s^2 + 0.352s + 0.392}$$

$$H(s_n) = \frac{0.081}{s^5 + 0.7048s^4 + 1.496s^3 + 0.689s^2 + 0.456s + 0.081}$$

Unnomalized transfer function, H(s) and $\Omega_p = 1\ rad/sec$
Hence

$$H(s_n) = \frac{0.081}{s^5 + 0.7048s^4 + 1.496s^3 + 0.689s^2 + 0.456s + 0.081}$$

Dec 2014: Design A Chebyshev I low pass filter that has to meet the following specifications

i) passband ripple $\leq$2 dB and passband edge frequency 1 rad/sec

ii) Stopband attenuation $\geq$20 dB and Stopband edge frequency 1.3 rad/sec

Solution:

Ω_p=1 rad/sec, Ω_s=1.3 rad/sec,
K_p=–2 dB, K_s=–20 dB,

$$K_p = 20\ log\left[\frac{1}{\sqrt{1+\epsilon^2}}\right] = -2$$

ϵ=0.76478

$$\delta_p = 1 - \frac{1}{\sqrt{1+\epsilon^2}} = 0.20567$$

$$K_s = 20\ log\delta_s = -20$$

δ_s=0.1

$$d = \sqrt{\frac{(1-\delta_p)^{-2}-1}{\delta_s^{-2}-1}} = 0.077$$

$$K = \frac{\Omega_p}{\Omega_s} = \frac{1}{1.3} = 0.769$$

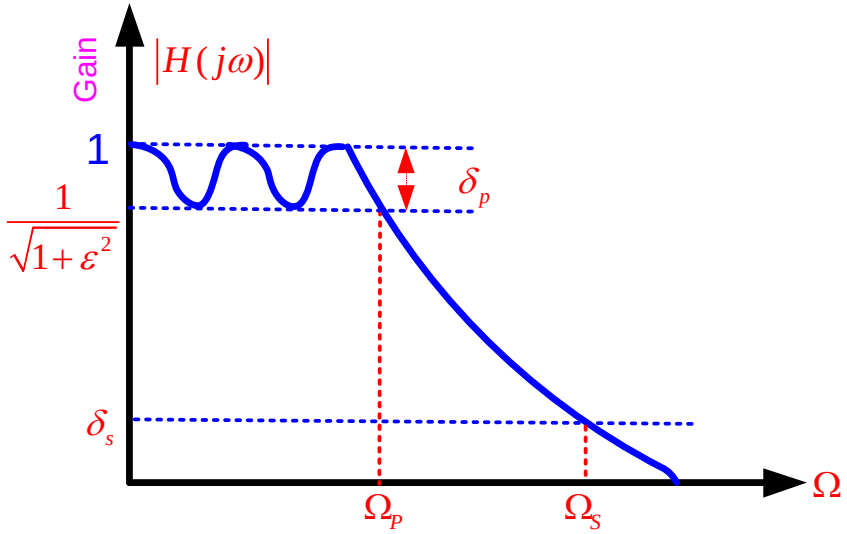


Figure 18: LPF specifications

The order of the filter is

$$N = \frac{cosh^{-1}\left(\frac{1}{d}\right)}{cosh^{-1}\left(\frac{1}{K}\right)} = 4.3 \simeq 5$$

$$a = \frac{1}{2}\left(\frac{1+\sqrt{1+\epsilon^2}}{\epsilon}\right)^{\frac{1}{N}} - \frac{1}{2}\left(\frac{1+\sqrt{1+\epsilon^2}}{\epsilon}\right)^{-\frac{1}{N}} = 0.21830398$$

$$b = \frac{1}{2}\left(\frac{1+\sqrt{1+\epsilon^2}}{\epsilon}\right)^{\frac{1}{N}} + \frac{1}{2}\left(\frac{1+\sqrt{1+\epsilon^2}}{\epsilon}\right)^{-\frac{1}{N}} = 1.0235520$$

$$\Omega_k = b\ cos\left[(2k-1)\frac{\pi}{2N}\right] = b\ cos\left[(2k-1)\frac{\pi}{10}\right]$$

$$\sigma_k = -a\ sin\left[(2k-1)\frac{\pi}{2N}\right] = -a\ sin\left[(2k-1)\frac{\pi}{10}\right]$$

where $k = 1, 2, \ldots 2N$ i.e., $k = 1, 2, \ldots 10$
The poles those are lie on left half of the s plane is

k	σ_k	Ω_k
1	-0.0674610	0.9734557
2	-0.1766151	0.6016287
3	-0.2183083	0
4	-0.1766151	-0.6016287
5	-0.0674610	-0.9734557

$$
\begin{aligned}
H_5(s) &= \frac{K_N}{\prod\limits_{LHPonly}(s-s_k)} = \frac{K_N}{(s-s_1)(s-s_2)(s-s_3)(s-s_4)} \\
&= \frac{K_N}{\begin{array}{l}(s+0.067461-j0.9734557)(s+0.067461+j0.9734557)\\ \quad (s+0.1766151-j0.6016287)(s+0.1766151+j0.6016287)(s+0.2180383)\end{array}} \\
&= \frac{K_N}{(s^2+0.134922s+0.95215)(s^2+0.35323s+0.393115)(s+0.2183083)} \\
&= \frac{K_N}{s^5+0.70646s^4+1.4995s^3+0.6934s^2+0.459349s+0.08172}
\end{aligned}
$$

N is odd $K_N = b_o$=0.08172

$$H_5(s) = \frac{0.08172}{s^5+0.70646s^4+1.4995s^3+0.6934s^2+0.459349s+0.08172}$$

Verification

$$
\begin{aligned}
H_a(j\Omega) &= \frac{0.08172}{(j\Omega)^5+0.70646(j\Omega)^4-1.49(j\Omega)^3-0.693(j\Omega)^2+0.4593(j\Omega)+0.08172} \\
|H_a(j\Omega)| &= \frac{0.08172}{\sqrt{(.7064\Omega^4-.693\Omega^2+.0817)^2+j(\Omega^5-1.499\Omega^3+.4593\Omega)^2}} \\
20\ log|H_a(j\Omega)|_1 &= -2\ dB \\
20\ log|H_a(j\Omega)|_1.3 &= -24.5\ dB
\end{aligned}
$$

Analog to analog frequency transformations

Design steps for highpass filter:

From the given specifications

1. Determine stopband frequency of the normalized lowpass filter by $\Omega_s = \frac{\Omega_p}{\Omega_s'}$
2. Determine the order of the Filter using

$$N = \frac{log\left[\frac{10^{\frac{-K_p}{10}}-1}{10^{\frac{-K_S}{10}}-1}\right]}{2log\left[\frac{\Omega_p}{\Omega_S}\right]}$$

3. Determine the cutoff frequency Ω_C using

$$\Omega_C = \frac{\Omega_s}{\left(10^{\frac{-K_s}{10}}-1\right)^{\frac{1}{2N}}} \quad OR \quad \Omega_C = \frac{\Omega_p}{\left(10^{\frac{-K_p}{10}}-1\right)^{\frac{1}{2N}}}$$

4. Determine the transfer function of normalized Butterworth filter by

$$H_N(s) = \frac{1}{\prod_{LHP}(s-s_k)} = \frac{1}{B_N(s)}$$

5. From analog lowpass to high frequency transformation, find the desired transfer function by substituting the following

$$H_a(s) = H_N(s)\big|_{s \to \frac{\Omega_p}{\Omega_c S}}$$

Design a Butterworth analog highpass filter that will meet the following specifications

i) Maximum passband attenuation gain 2 dB
ii) Passband edge frequency=200 rad/sec
iii) Minimum stopband attenuation =20 dB
iv) Stopband edge frequency=100 rad/sec

Determine the transfer function $H_a(s)$ of the lowest order Butterworth filter to meet the above the specifications

Solution:

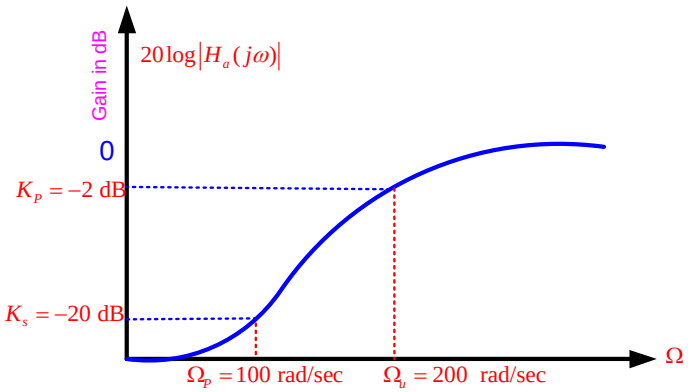


Figure 19: HPF specifications

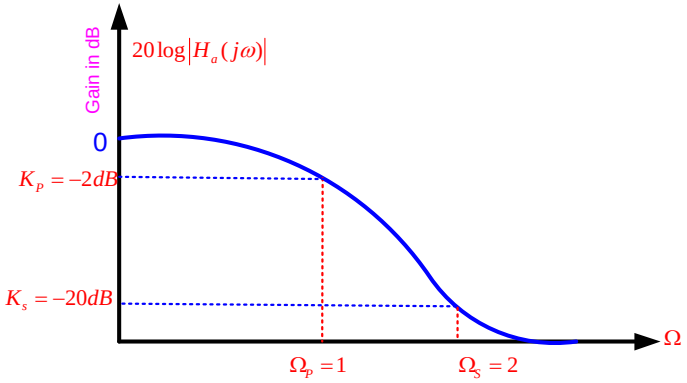


Figure 20: normalized LPF specifications

$$\Omega_S = \frac{\Omega_p}{\Omega'_S} = \frac{200}{100} = 2$$

$$\Omega_S = 2 \qquad \Omega_p = 1$$

The order of the filter is

$$N = \frac{log\left[\frac{10^{\frac{-K_p}{10}}-1}{10^{\frac{-K_S}{10}}-1}\right]}{2log\left[\frac{\Omega_p}{\Omega_S}\right]} = 3.7 \simeq 4$$

For Even N=4

$$H(s_n) = \prod_{k=1}^{\frac{N}{2}} \frac{1}{s^2 + b_k s + 1}$$

where $b_k = 2sin\left[\frac{(2k-1)\pi}{2N}\right]$

N=4

$k = \frac{N}{2} = \frac{4}{2} = 2$

k=1

$b_k = b_1 = 2sin\left[\frac{(2-1)\pi}{2\times 4}\right]$=0.7654

k=2

$b_k = b_2 = 2sin\left[\frac{(4-1)\pi}{2\times 4}\right]$=1.8478

$$\begin{aligned} H(s_n) &= \frac{1}{(s^2 + 0.7654s + 1)(s^2 + 1.8478s + 1)} \\ &= \frac{1}{s^4 + 2.6118s^3 + 3.4117s^2 + 2.6118s + 1} \end{aligned}$$

$$\begin{aligned} \Omega_c &= \frac{\Omega_s}{(10^{\frac{-k_s}{10}} - 1)^{\frac{1}{2N}}} \\ &= \frac{2}{(10^{\frac{20}{10}} - 1)^{\frac{1}{8}}} = \frac{2}{1.776} \\ &= 1.126\ rad/sec \end{aligned}$$

$$
\begin{aligned}
H_a(s) &= \frac{1}{(s^2+0.7654s+1)(s^2+1.8478s+1)} \\
H_a(s) &= H_p(s)\big|_{s\rightarrow \frac{\Omega_p}{\Omega_c s}} \\
&= H_4(s)\big|_{s\rightarrow \frac{200}{1.126s}} \\
&= H_4(s)\big|_{s\rightarrow \frac{177.62}{s}} \\
&= \frac{s^4}{(s^2+135.95s+31548.86)(s^4+328.206s+31548.86)}
\end{aligned}
$$

Verification

$$
\begin{aligned}
H_a(j\Omega) &= \frac{\Omega^4}{\begin{array}{c}[(34980.7521-\Omega^2)+j1431.1464\Omega] \\ [(34980.7521-\Omega^2)+j1431.1464\Omega]\end{array}} \\
|H_a(j\Omega)| &= \frac{\Omega^4}{\begin{array}{c}\sqrt{[(34980.7521-\Omega^2)^2+(1431.1464\Omega)^2]} \\ \sqrt{[(34980.7521-\Omega^2)^2+(1431.1464\Omega)^2]}\end{array}} \\
20\ log|H_a(j\Omega)|_{\Omega=200} &= -2\ dB \\
20\ log|H_a(j\Omega)|_{\Omega=100} &= -21.83\ dB
\end{aligned}
$$

Design a Butterworth analog bandpass filter that will meet the following specifications

i) a -3.0103 dB upper and lower cutoff frequency of 50 Hz and 20 KHz
ii) a Stopband attenuation of atleast 20 dB at 20 Hz and 45 KHz and
iii) Monotonic frequency response.

Solution:

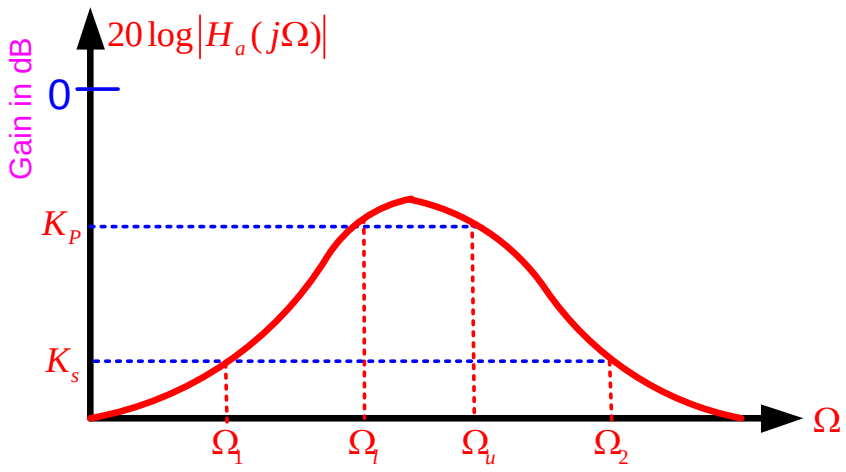


Figure 21: HPF specifications

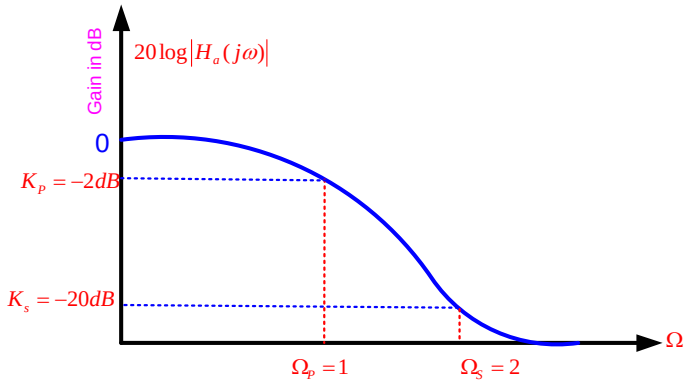


Figure 22: normalized LPF specifications

$$\Omega_1 = 2\pi \times 20 = 125.663 rad/sec$$
$$\Omega_2 = 2\pi \times 45 \times 10^3 = 2.827 \times 10^5 rad/sec$$
$$\Omega_u = 2\pi \times 20 \times 10^3 = 1.257 \times 10^5 rad/sec$$
$$\Omega_l = 2\pi \times 50 \times 10^3 = 314.159 \times rad/sec$$

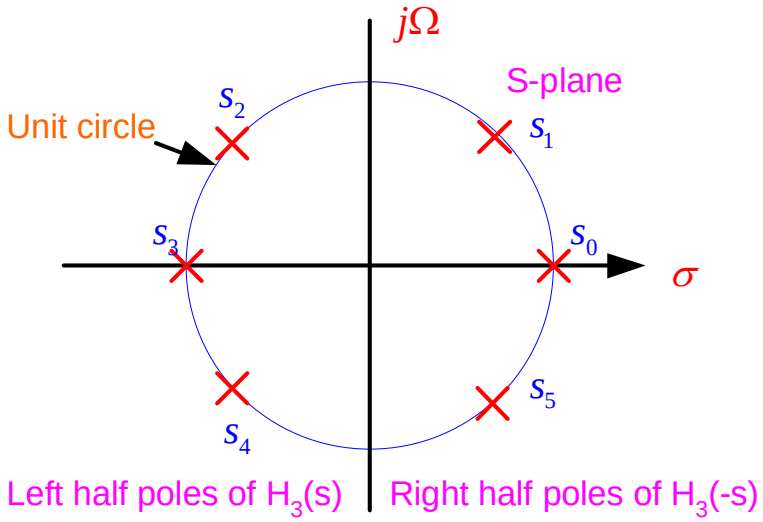
$$A = \frac{-\Omega^2{}_l + \Omega_l\Omega_u}{\Omega_l(\Omega_u - \Omega_l)} = 2.51$$

$$B = \frac{-\Omega^2{}_2 - \Omega_l\Omega_u}{\Omega_2(\Omega_u - \Omega_l)} = 2.25$$

$$\Omega_S = Min[|A|, |B|] = 2.25$$

The order of the filter is

$$N = \frac{log\left[\frac{10^{\frac{-K_p}{10}}-1}{10^{\frac{-K_S}{10}}-1}\right]}{2log\left[\frac{\Omega_p}{\Omega_S}\right]} = 2.83 \simeq 3$$



For odd N=3

$$H(s_n) = \frac{1}{(s+1)}\prod_{k=1}^{\frac{N-1}{2}}\frac{1}{s^2+b_k s+1}$$

where $b_k = 2sin\left[\frac{(2k-1)\pi}{2N}\right]$
N=3
$k = \frac{N-1}{2} = \frac{3-1}{2} = 1$
k=1
$b_k = b_1 = 2sin\left[\frac{(2-1)\pi}{2\times 3}\right]$=1

$$\begin{aligned} H(s_n) &= \frac{1}{(s+1)(s^2+s+1)} \\ &= \frac{1}{s^3+2s^2+2s+1} \end{aligned}$$

$$\begin{aligned} H_a(s) &= H_3(s)\Big|_{s\rightarrow\frac{s^2+\Omega_u\Omega_l}{s(\Omega_u-\Omega_l)}} \\ &= H_3(s)\Big|_{s\rightarrow\frac{s^2+3.949\times 10^7}{s(1.2538\times 10^5)}} \end{aligned}$$

Let $H(s) = \frac{1}{s^2+s+1}$ represent the transfer function of LPF with passband of 1 rad/sec. Use frequency transformation to find the transfer functions of the following analog filters

i) A lowpass filter with a passband of 10 rad/sec
ii) A highpass filter with a cutoff frequency of 1 rad/sec
iii) A highpass filter with a cutoff frequency of 10 rad/sec
iv) A bandpass filter with a passband of 10 rad/sec and a center frequency of 100 rad/sed
v) A bandstop filter with a stopband of 2 rad/sec and a center frequency of 10 rad/sed

Solution: Lowpass to highpass transformation

$$H(s) = \frac{1}{(s^2+s+1)}$$

$$\begin{aligned} H_a(s) &= H_3(s)|_{s \to \frac{s}{10}} \\ &= \frac{1}{\left(\frac{s}{10}\right)^2 + \left(\frac{s}{10}\right) + 1} = \frac{100}{s^2+10s+100} \end{aligned}$$

Lowpass to highpass transformation

$$\begin{aligned} H_a(s) &= H_3(s)|_{s \to \frac{\Omega_p}{\Omega_c s}} = \frac{10}{1s} \\ &= \frac{1}{\left(\frac{10}{5}\right)^2 + \left(\frac{10}{s}\right) + 1} = \frac{s^2}{s^2+10s+100} \end{aligned}$$

Lowpass to bandpass transformation

$$s \rightarrow \frac{s^2 + \Omega_u \Omega_l}{s(\Omega_u - \Omega_l)} = \frac{s + \Omega_0^2}{s + B_0}$$

where $\Omega_0 = \sqrt{\Omega_u \Omega_l}$ and $B_0 = \Omega_u \Omega_l$

$$\begin{aligned} H_a(s) &= H(s)|_{s \rightarrow \frac{s^2 + 10 \times 10^4}{10s}} \\ &= = \frac{100s^2}{s^4 + 10s^3 + 20100s^2 + 10^4 s + 10^8} \end{aligned}$$

Lowpass to bandstop transformation

$$s \rightarrow \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u \Omega_l} = \frac{sB_0}{s^2 + \Omega^2}$$

$$\begin{aligned} H_a(s) &= H(s)|_{s \rightarrow \frac{2s}{s^2 + 100}} \\ &= = \frac{(s^2 + 100)^2}{s^4 + 2s^3 + 204s^2 + 200s + 10^4} \end{aligned}$$

Table 2: Comparison between Butterworth and Chebyshev Filter

	Butterworth Filter	Chebyshev Filter
1	The magnitude frequency response is montonically decreasing	The magnitude frequency response has ripples in passband or stopband
2	The poles lie on a circle in the s plane	The poles lie on an ellipse in the s plane
3	For a given frequency specifications the number of poles are more	For a given frequency specifications the number of poles are less
4	For a given order N the width of the transition band is more	For a given order N the width of the transition band is less
5	Only few parameters has to be calculated to determine the transfer function	A large number of parameters has to be calculated to determine the transfer function

Table 3: Comparison between IIR and FIR Filter

	IIR Filter	FIR Filter
1	Linear characteristic cannot be achieved	Linear characteristic can be achieved
2	The impulse response cannot be directly converted to digital filter transfer function	The impulse response can be directly converted to digital filter transfer function
3	It is recursive filter and may be stable or unstable	It may be recursive or non recursive filter and recursive filter are stable
4	The specifications include the desired characteristics for magnitude response only	The specifications include the desired characteristics for both magnitude and phase response
5	The design involves design of analog filter and then transforming analog to digital filter	The digital filter can be directly designed to achieve the desired specifications.

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