Chapter 1

Laplace Transform

If f(t) is a function in time, and its Laplace transform F(s) is expressed as:

$$L[f(t)] = F(S) = \int_0^\infty f(t)e^{-st}dt \qquad (1.1)$$

Q 1) Find the Laplace transform of the unit step function as shown in Figure 1.2.

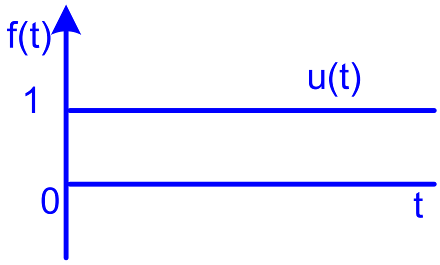

Figure 1.1: Step function

Solution:

The unit step function is defined as

$$u(t) = \begin{cases} 1, & for\ t \geq 0 \\ 0, & for\ t < 0 \end{cases}$$

$$\begin{aligned} L[f(t)] &= \int_0^\infty f(t)e^{-st}dt = \int_0^\infty 1e^{-st}dt \\ &= \left[-\frac{1}{S}e^{-st}\right]_0^\infty = \frac{1}{S} \end{aligned}$$

Q 2) Find the Laplace transform of the $f(t) = e^{at}$ where a is constant.

Solution:

$$\begin{aligned} L[f(t)] &= \int_0^\infty f(t)e^{-st}dt = \int_0^\infty e^{at}e^{-st}dt \\ &= \int_0^\infty e^{-(s-a)t}dt \left[-\frac{1}{s-a}e^{-st}\right]_0^\infty = \frac{1}{s-a} \end{aligned}$$

Similarly Laplace transform of the $f(t) = e^{-at}$ is

$$\begin{aligned} L[f(t)] &= \int_0^\infty f(t)e^{-st}dt = \int_0^\infty e^{-at}e^{-st}dt \\ &= \int_0^\infty e^{-(s+a)t}dt \left[-\frac{1}{s+a}e^{-st}\right]_0^\infty = \frac{1}{s+a} \end{aligned}$$

Q 3) Find the Laplace transform of the $f(t) = sin\omega t$.

Solution:

$$\begin{aligned} L[f(t)] &= \int_0^\infty f(t)e^{-st}dt = \int_0^\infty sin\omega te^{-st}dt \\ &= \int_0^\infty \frac{e^{j\omega t} - e^{-j\omega t}}{2j}e^{-st}dt \\ &= \frac{1}{2j}\left[\int_0^\infty (e^{-(s-j\omega t)}dt - \int_0^\infty (e^{-(s+j\omega t)}dt\right] \\ &= \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right] = \frac{1}{2j}\frac{2j\omega}{s^2+\omega^2} \\ &= \frac{\omega}{s^2+\omega^2} \end{aligned}$$

Q 4) Find the Laplace transform of the $f(t) = cos\omega t$.

Solution:

$$
\begin{aligned}
L[f(t)] &= \int_0^\infty f(t)e^{-st}dt = \int_0^\infty cos\omega t e^{-st}dt \\
&= \int_0^\infty \frac{e^{j\omega t}+e^{-j\omega t}}{2}e^{-st}dt \\
&= \frac{1}{2j}\left[\int_0^\infty (e^{-(s-j\omega t)}dt + \int_0^\infty (e^{-(s+j\omega t)}dt\right] \\
&= \frac{1}{2}\left[\frac{1}{s-j\omega}+\frac{1}{s+j\omega}\right] = \frac{1}{2}\frac{2s}{s^2+\omega^2} \\
&= \frac{s}{s^2+\omega^2}
\end{aligned}
$$

Q 5 2011-DEC-7a) Find the Laplace transform i) sin^2t ii) cos^2t, iii) $sin\omega t$

i) sin^2t

$$
\begin{aligned}
f(t) &= sin^2t \\
f(t) &= \left[\frac{1-cos2t}{2}\right] = \frac{1}{2}-\frac{1}{2}cos2t \\
F(s) &= \frac{1}{2s}-\frac{1}{2}\left[\frac{s}{s^2+4}\right] \\
&= \frac{s^2+4-s^2}{2s(s^2+4)} \\
&= \frac{2}{s(s^2+4)}
\end{aligned}
$$

ii) cos^2t

$$
\begin{aligned}
f(t) &= cos^2t \\
f(t) &= \left[\frac{1+cos2t}{2}\right] = \frac{1}{2}+\frac{1}{2}cos2t \\
F(s) &= \frac{1}{2s}+\frac{1}{2}\left[\frac{s}{s^2+4}\right] \\
&= \frac{s^2+4+s^2}{2s(s^2+4)} \\
&= \frac{2(s^2+2)}{2s(s^2+4)} \\
&= \frac{s^2+2}{s(s^2+4)}
\end{aligned}
$$

Time Shifting Property

If L[f(t)]=F(s) then, for any real number t_o

$$L[f(t-t_o)u(t-t_o)] = e^{-t_os}F(s)$$

$$
\begin{aligned}
L[f(t-t_o)u(t-t_o)] &= \int_0^\infty f(t)e^{-st}dt \\
&= \int_0^\infty f(t-t_o)u(t-t_o)e^{-st}dt
\end{aligned}
$$

$$u(t-t_o) = \begin{cases} 1, & for\ t \geq t_o \\ 0, & for\ t < t_o \end{cases}$$

$$L[f(t-t_o)u(t-t_o)] = \int_{t_o}^\infty f(t-t_o)e^{-st}dt$$

Let $\tau = t-t_o \quad t = \tau + t_o$

$$
\begin{aligned}
L[f(t-t_o)u(t-t_o)] &= \int_0^\infty f(\tau)e^{-s(\tau+t_o)}d\tau \\
&= e^{-t_os}\int_0^\infty f(\tau)e^{-s\tau}d\tau \\
&= e^{-t_os}F(s)
\end{aligned}
$$

Differentiation in Time domain

If L[f(t)]=F(s) then

$$L\left[\frac{df(t)}{dt}\right] = sF(s)-f(0)$$

Let

$$y(t) = \frac{df(t)}{dt}$$

$$L[y(t)] = \int_0^\infty y(t)e^{-st}dt$$

$$Y[s] = \int_0^\infty \frac{df(t)}{dt}e^{-st}dt$$

Integrating by parts

$$
\begin{aligned}
Y[s] &= \left[e^{-st}f(t)\right]_0^\infty - \int_0^\infty f(t)(-se^{-st})dt \\
&= 0-f(0)+s\int_0^\infty f(t)e^{-st}dt \\
&= -f(0)+sF(s) \\
&= sF(s)-f(0)
\end{aligned}
$$

Similarly

$$L\left[\frac{d^nf(t)}{dt^n}\right] = s^nF(s)-s^{n-1}f(0)\cdots - f^{n-1}(0)$$

Integration in Time domain

If

$$y(t) = \int_0^t f(\tau)d\tau$$

then

$$L[y(t)] = Y(s) = \frac{F(s)}{s}$$

$$L[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt$$

Divide both sides by s

$$\frac{F(s)}{s} = \int_0^\infty f(t)\frac{e^{-st}}{s}dt$$

$$\begin{aligned}\frac{F(s)}{s} &= \left[\frac{e^{-st}}{s}y(t)\right]_0^\infty - \int_0^\infty y(t)\frac{e^{-st}}{s}(-s)dt \\ &= y(t)\left[\frac{e^{-st}}{s}\right]_0^\infty + \int_0^\infty y(t)\frac{e^{-st}}{s}(s)dt\end{aligned}$$

At $t = \infty$

$$e^{-\infty} = 0$$

and

$$\begin{aligned}y(t) &= \int_0^t f(\tau)d\tau \\ y(0) &= \int_0^0 f(\tau)d\tau = 0\end{aligned}$$

Hence

$$Y(s) = \frac{F(s)}{s}$$

Table 1.1: Laplace Transform

Function	Laplace Transform	Function	Laplace Transform
u(t)	$\frac{1}{S}$	$\delta(t)$	1
t	$\frac{1}{S^2}$	t^n	$\frac{n!}{S^{n+1}}$
e^{at}	$\frac{1}{S-a}$	e^{-at}	$\frac{1}{S+a}$
te^{-at}	$\frac{1}{(S+a)^2}$	te^{+at}	$\frac{1}{S-a)^2}$
$e^{at}f(t)$	$F(S-a)$	$e^{-at}f(t)$	$F(S+a)$
$sin\omega t$	$\frac{\omega}{S^2+\omega^2}$	$cos\omega t$	$\frac{S}{S^2+\omega^2}$
$e^{-at}sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

Q 1) Find the Laplace transform of the unit step function as shown in Figure 1.2.

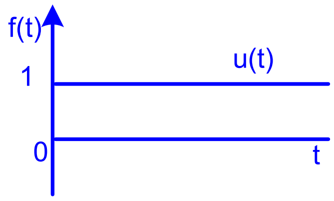

Figure 1.2: Step function

Solution:

The unit step function is defined as

$$u(t) = \begin{cases} 1, & for\ t \geq 0 \\ 0, & for\ t < 0 \end{cases}$$

$$\begin{aligned} L[f(t)] &= \int_0^\infty f(t)e^{-st}dt = \int_0^\infty 1e^{-st}dt \\ F(s) &= \left[-\frac{1}{s}e^{-st}\right]_0^\infty = \frac{1}{s} \end{aligned}$$

Q 2) Find the Laplace transform of the unit step function u(t-1) as shown in Figure 1.3 (a).

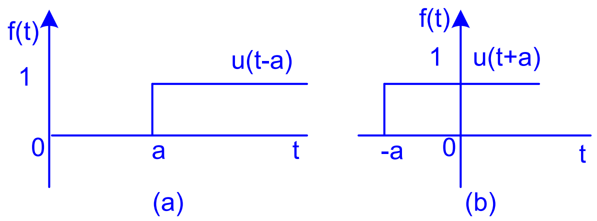

Figure 1.3: Step function

Solution:

The unit step function is defined as

$$u(t-a) = \begin{cases} 1, & for\ t \geq a \\ 0, & for\ t < a \end{cases}$$

Using Time shift property

$$L[f(t)] = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty u(t-a)e^{-st}dt = \frac{1}{s}e^{-as}$$

Q 3) Find the Laplace transform of the unit step function u(t+1) as shown in Figure 1.3 (b).

The unit step function is defined as

$$u(t+a) = \begin{cases} 1, & for\ t \geq -a \\ 0, & for\ t < -a \end{cases}$$

Using Time shift property

$$L[f(t)] = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty u(t+a)e^{-st}dt = \frac{1}{S}e^{as}$$

Q 4) Find the Laplace transform of the unit ramp function r(t) as shown in Figure 1.3 (b).

The ramp function is defined as

$$r(t) = \begin{cases} 0, & t \leq 0 \\ t, & t \geq 0 \end{cases}$$

$$L[f(t)] = F[s] = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty te^{-st}dt$$

Integrating by parts and substituting limits

$$\begin{aligned} \int_0^\infty te^{-st}dt &= \left[\frac{te^{-st}}{-s}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}dt \\ &= 0 + \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s^2} \end{aligned}$$

Q 5) Find the Laplace transform of the unit ramp function r(t) as shown in Figure 1.4.

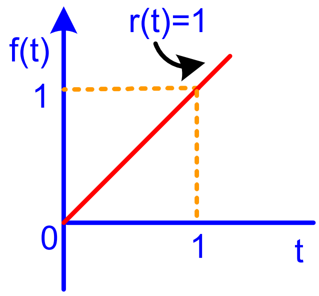

Figure 1.4: Unit ramp function

The ramp function is defined as

$$r(t) = \begin{cases} t, & t \geq a \\ 0, & t \leq 0 \end{cases}$$

$$L[f(t)] = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty te^{-st}dt = \frac{1}{s^2}$$

Q 6) Find the Laplace transform of the unit ramp function r(t) as shown in Figure 1.6 (a).

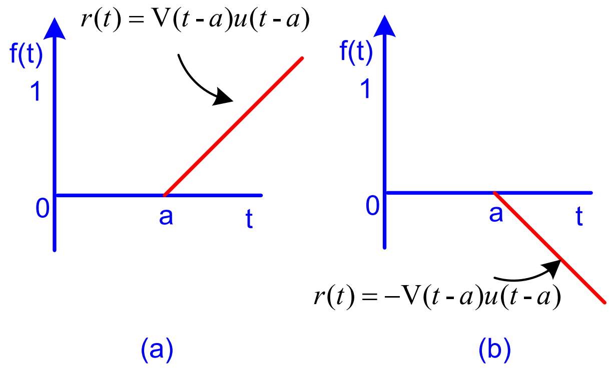

Figure 1.5: Unit ramp function

The ramp function is defined as

$$Ar(t-a) = \begin{cases} (t-a), & t \geq a \\ 0, & t \leq 0 \end{cases}$$

$$L[f(t)] = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty (t-a)e^{-st}dt = \frac{A}{s^2}e^{-as}$$

Q 7) Find the Laplace transform of the unit ramp function r(t) as shown in Figure 1.6 (b).

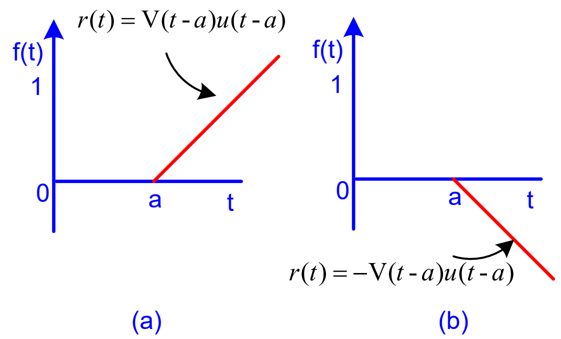

Figure 1.6: Unit ramp function

The ramp function is defined as

$$r(t) = \begin{cases} -A(t+a), & t \geq a \\ 0, & t \leq 0 \end{cases}$$

$$L[f(t)] = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty (t-a)e^{-st}dt = -\frac{A}{s^2}e^{-as}$$

Q 8) Find the Laplace transform of the stair case waveform as shown in Figure 1.7.

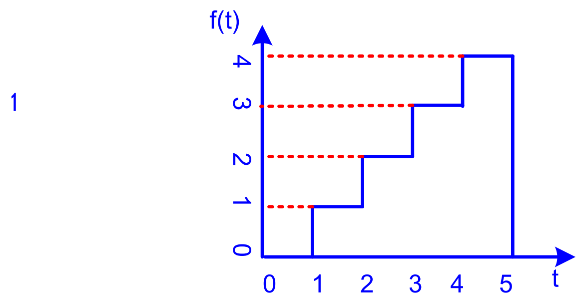

Figure 1.7: Staircase function

The ramp function is defined as

$$u(t) = \begin{cases} 1, & 1 < t < 2 \\ 2, & 2 < t < 3 \\ 3, & 3 < t < 4 \\ 4, & 4 < t < 5 \\ 0, & otherwis3 \end{cases}$$

$$u(t) = [u(t-1) - u(t-2)] + 2[u(t-2) - u(t-3)]$$

$$\begin{aligned} u(t) &= [u(t-1) - u(t-2)] + 2[u(t-2) - u(t-3)] \\ &+ 3[u(t-3) - u(t-4)] + 4[u(t-4) - u(t-5)] \\ &= u(t-1) + u(t-2) + u(t-3) + u(t-4) \\ &\quad -4u(t-5) \end{aligned}$$

$$U(s) = \frac{1}{s}[e^{-s} + e^{-2s} + e^{-3s} + e^{-4s} - 4e^{-5s}]$$

Q 1) Find the Laplace transform of the square wave as shown in Figure 1.8.

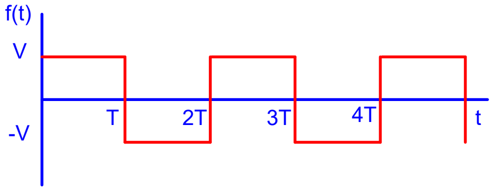

Figure 1.8: Square wave

Solution:

By considering one complete cycle from 0 to 2T

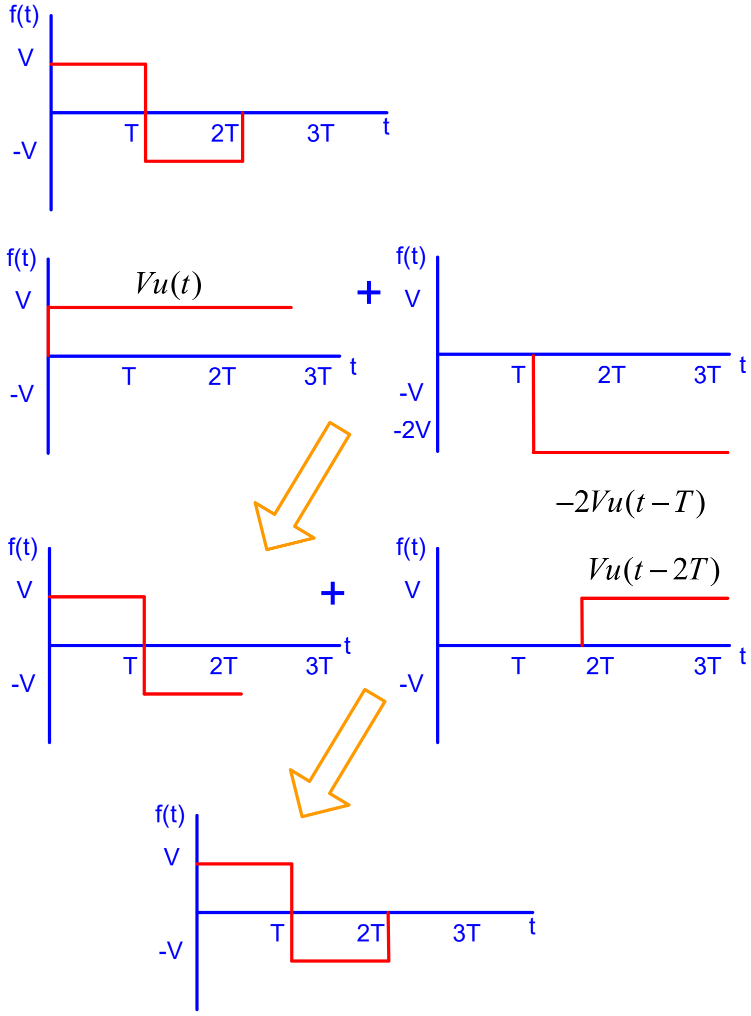

Figure 1.9: Square wave

$$
\begin{aligned}
f_1(t) &= Vu(t) - 2Vu(t-T) + Vu(t-2T) \\
F_1(S) &= V\frac{1}{s} - 2V\frac{1}{s}e^{-TS} + Ve^{-2TS}\frac{1}{s} \\
&= \frac{V}{s}\left[1 - 2e^{-TS} + e^{-2TS}\right] \\
&= \frac{V}{s}\left[1 - e^{-TS}\right]^2
\end{aligned}
$$

Laplace transform for complete periodic waveform is

$$
\begin{aligned}
L[f(t)] &= \left[\frac{1}{1-e^{-st)}}\right] F_1(S) \\
L[f(t)] &= \left[\frac{1}{1-e^{-s2T)}}\right] \frac{V}{S}\left[(1-e^{-sT})^2\right] \\
&= \frac{V}{S}\left[\frac{(1-e^{-sT})^2}{1-(e^{-sT})^2}\right] \\
&= \frac{V}{S}\left[\frac{(1-e^{-sT})^2}{(1+(e^{-sT})(1-(e^{-sT})}\right] \\
&= \frac{V}{S}\left[\frac{(1-e^{-sT})}{(1+e^{-sT})}\right]
\end{aligned}
$$

Q 2) Find the Laplace transform of the waveform as shown in Figure 1.10.

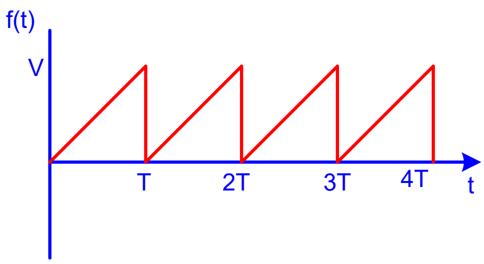

Figure 1.10

Solution:

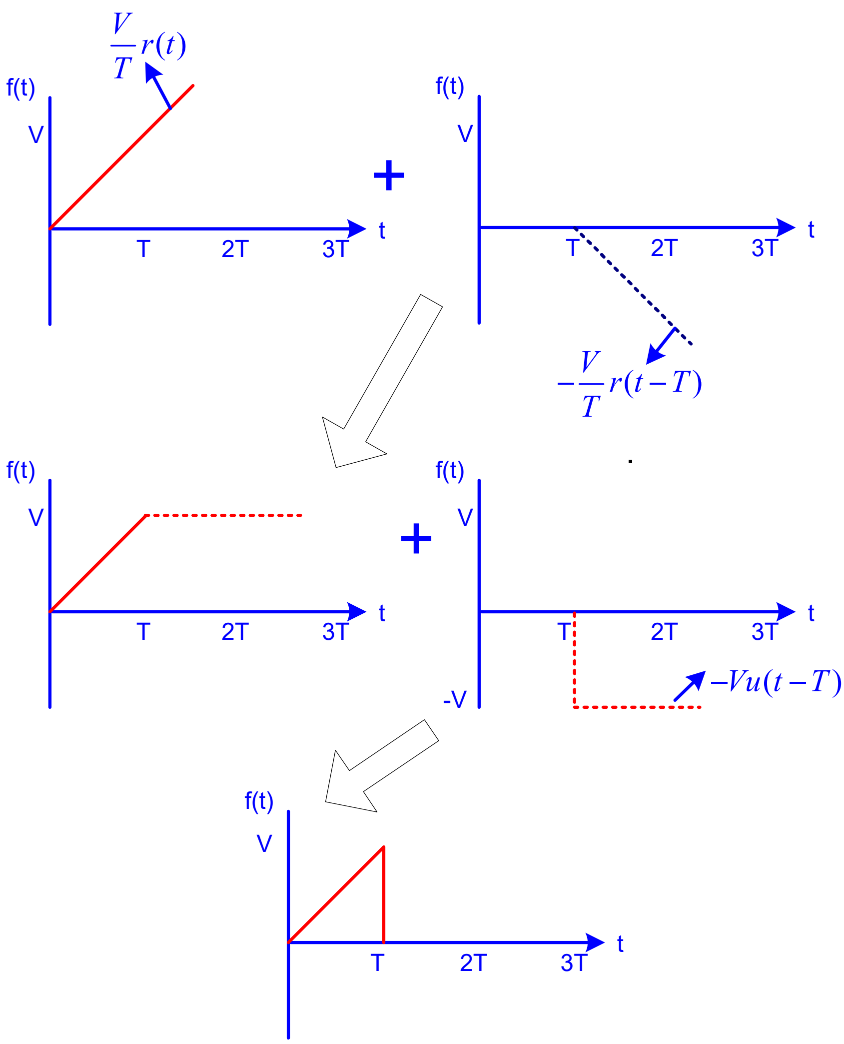

Figure 1.11

By considering one complete cycle from 0 to T

$$
\begin{aligned}
L[f_(t)] &= \frac{V}{T}r(t) - \frac{V}{T}r(t-T) - Vu(t-T) \\
F_1(S) &= \frac{V}{T}\frac{1}{S^2} - \frac{V}{T}\left[e^{-TS}\frac{1}{S^2}\right] - V\frac{1}{S}e^{-TS} \\
&= \frac{V}{TS^2}\left[1 - e^{-TS} - TSe^{-TS}\right]
\end{aligned}
$$

Laplace transform of the complete periodic waveform is

$$
\begin{aligned}
F(S) &= \frac{F_1(S)}{1-e^{-TS}} \\
&= \frac{V}{TS^2(1-e^{-TS})}\left[1 - e^{-TS} - TSe^{-TS}\right] \\
&= \frac{V}{TS^2(1-e^{-TS})}\left[1 - e^{-TS} - TSe^{-TS}\right]
\end{aligned}
$$

Q 3) Find the Laplace transform of the waveform as shown in Figure 1.12.

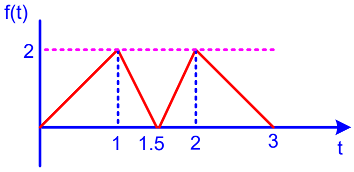

Figure 1.12

Solution:

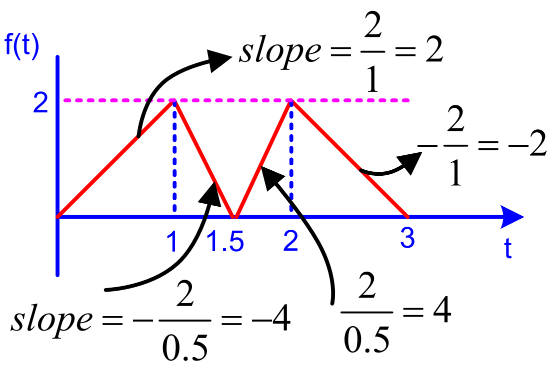

Figure 1.13

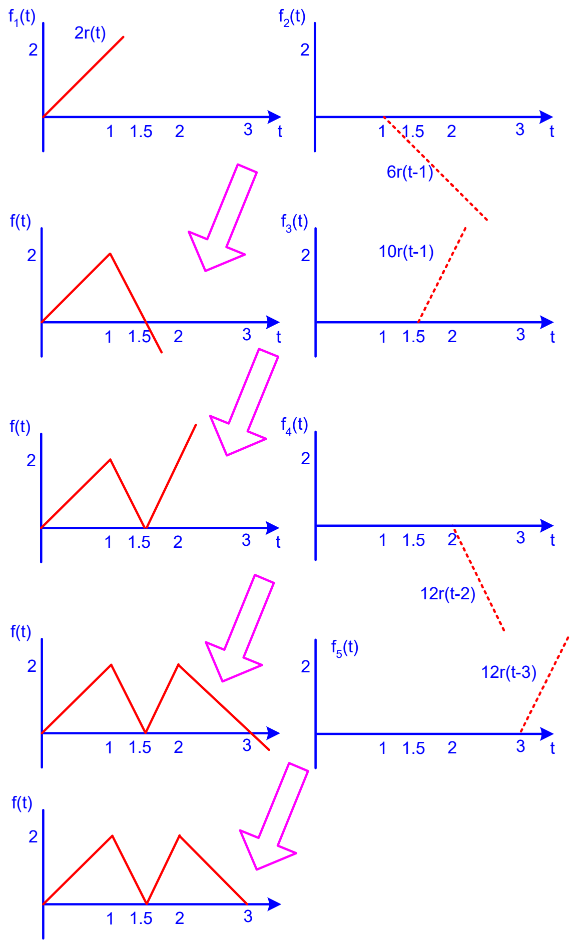

Figure 1.14

$$
\begin{aligned}
f(t) &= 2r(t) - 6r(t-1) + 10r(t-1.5) - 12r(t-2) \\
&\quad +12r(t-3) \\
F(S) &= \frac{2}{s^2} - \frac{6}{s^2}e^{-s} + \frac{10}{s^2}e^{-2s} - \frac{12}{s^2}e^{-2s} + \frac{12}{s^2}e^{-3s} \\
&= \frac{2}{s^2}\left[1 - 3e^{-S} + 5e^{-1.5s} - 6e^{-2s} + 6e^{-3s}\right]
\end{aligned}
$$

Q 4) Find the Laplace transform of the waveform as shown in Figure 1.15.

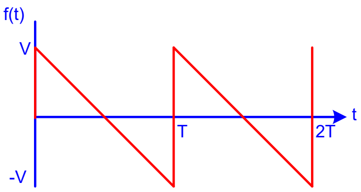

Figure 1.15: Sawtooth Wave

Solution:

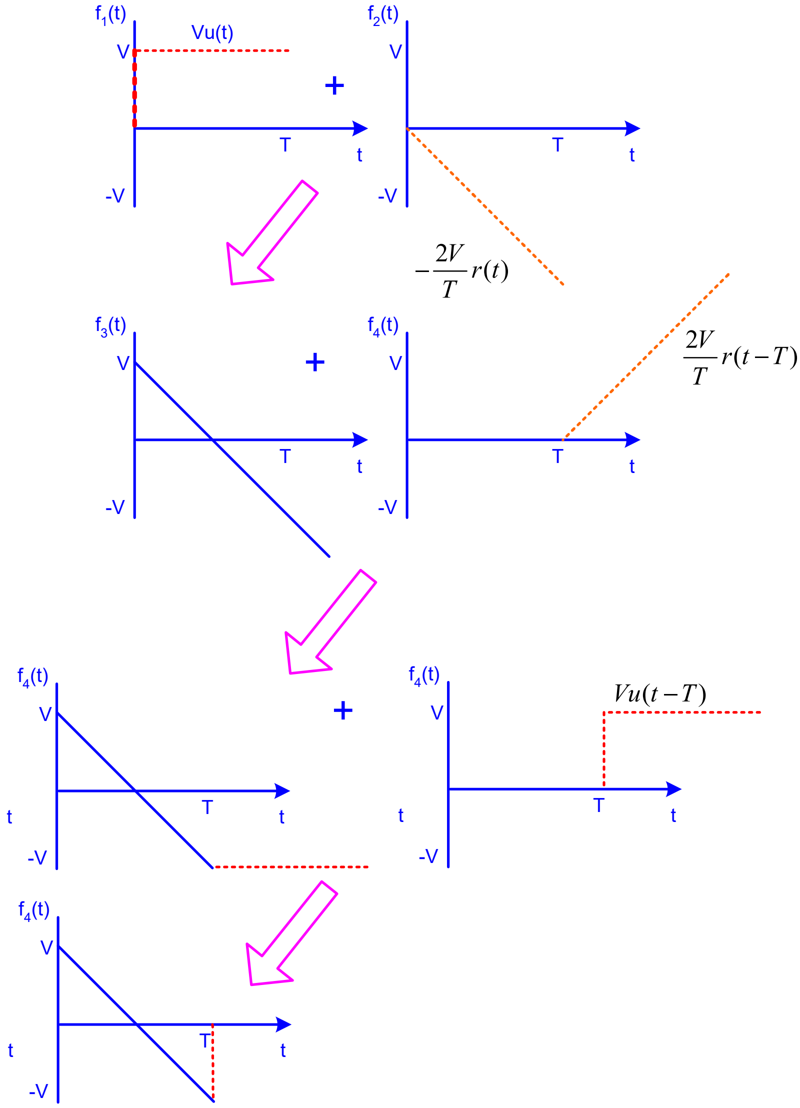

Figure 1.16: Sawtooth Wave

By considering one complete cycle from 0 to T

$$
\begin{aligned}
f(t) &= Vu(t) - \frac{2V}{T}r(t) + \frac{2V}{T}r(t-T) + Vu(t-T) \\
F_1(S) &= \frac{V}{s} - \frac{2V}{T}\frac{1}{S^2} + \frac{2V}{T}\left[e^{-TS}\frac{1}{S^2}\right] + \frac{V}{s}e^{-TS} \\
&= \frac{V}{s}\left[1+e^{-TS}\right] - \frac{2V}{TS^2}\left[1-e^{-TS}\right]
\end{aligned}
$$

Laplace transform of the periodic waveform is

$$
F(S) = \frac{F_1(S)}{1-e^{-TS}} = \frac{1}{1-e^{-TS}}\frac{V}{s}\left[1+e^{-TS}\right] - \frac{2V}{TS^2}\left[1-e^{-TS}\right]
$$

Q 5) Find the Laplace transform of the waveform as shown in Figure 1.17.

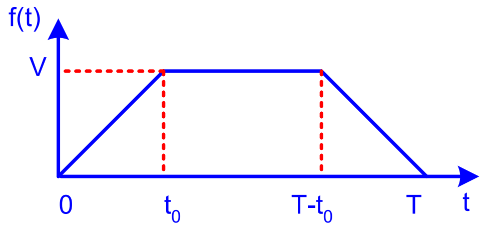

Figure 1.17: Trapezoid Wave

Solution:

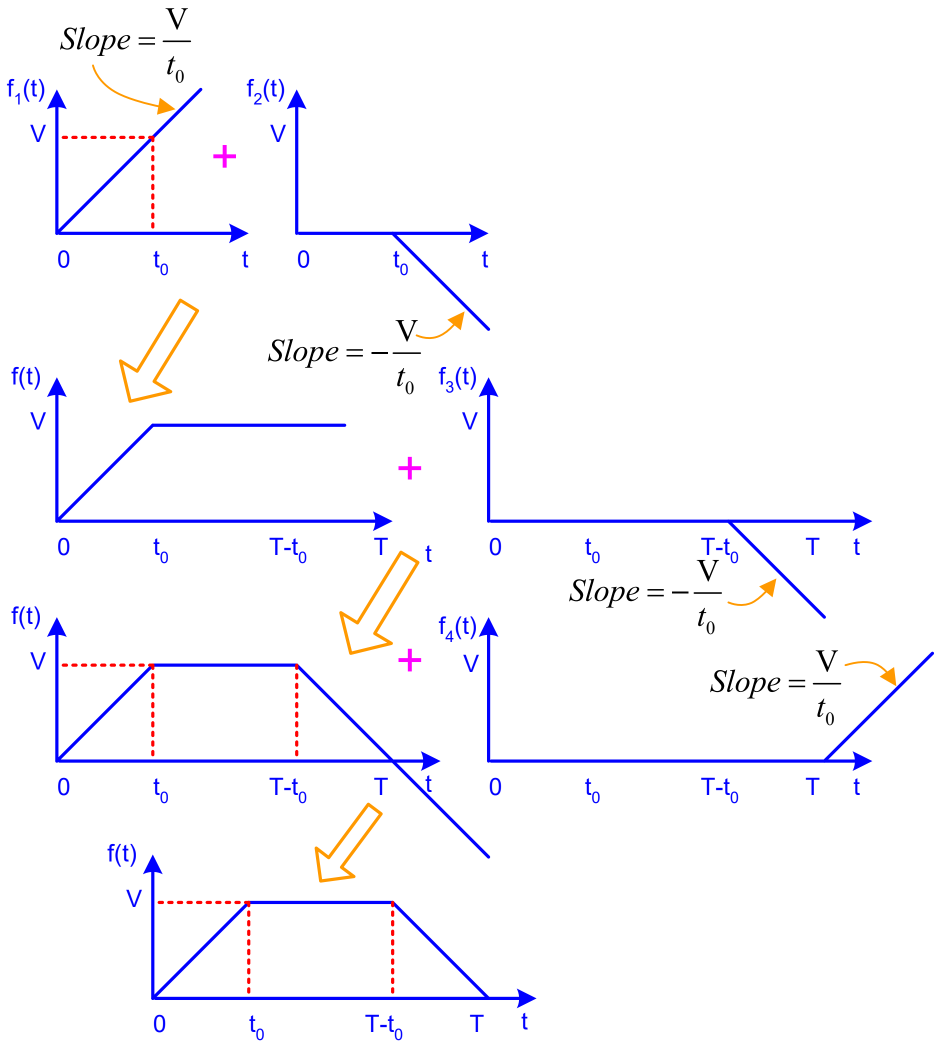

Figure 1.18: Trapezoid Wave

$$
\begin{aligned}
f(t) &= f_1(t) + f_2(t) + f_3(t) + f_4(t) \\
&= \frac{V}{t_0}tu(t) - \frac{V}{t_0}(t-t_0)u(t-t_0) - \frac{V}{t_0}[t-(T-t_0)-u(t-(T-t_0))] + \frac{V}{t_0}[t-T)-u(t-T)] \\
F(s) &= \frac{V}{t_0}\frac{1}{s^2} - \frac{V}{t_0}\frac{1}{s^2}e^{-t_0 s} - \frac{V}{t_0}\frac{1}{s^2}e^{-(T-t_0)s} + \frac{V}{t_0}\frac{1}{s^2}e^{-Ts}
\end{aligned}
$$

Q 6) Find the Laplace transform of the waveform as shown in Figure 1.19.

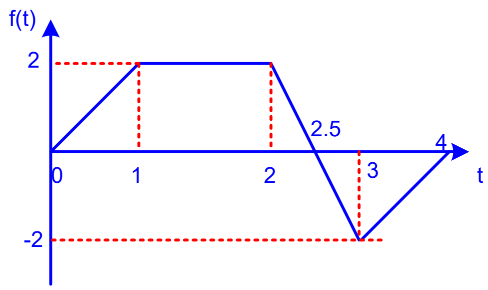

Figure 1.19: Trapezoid Wave

Solution:

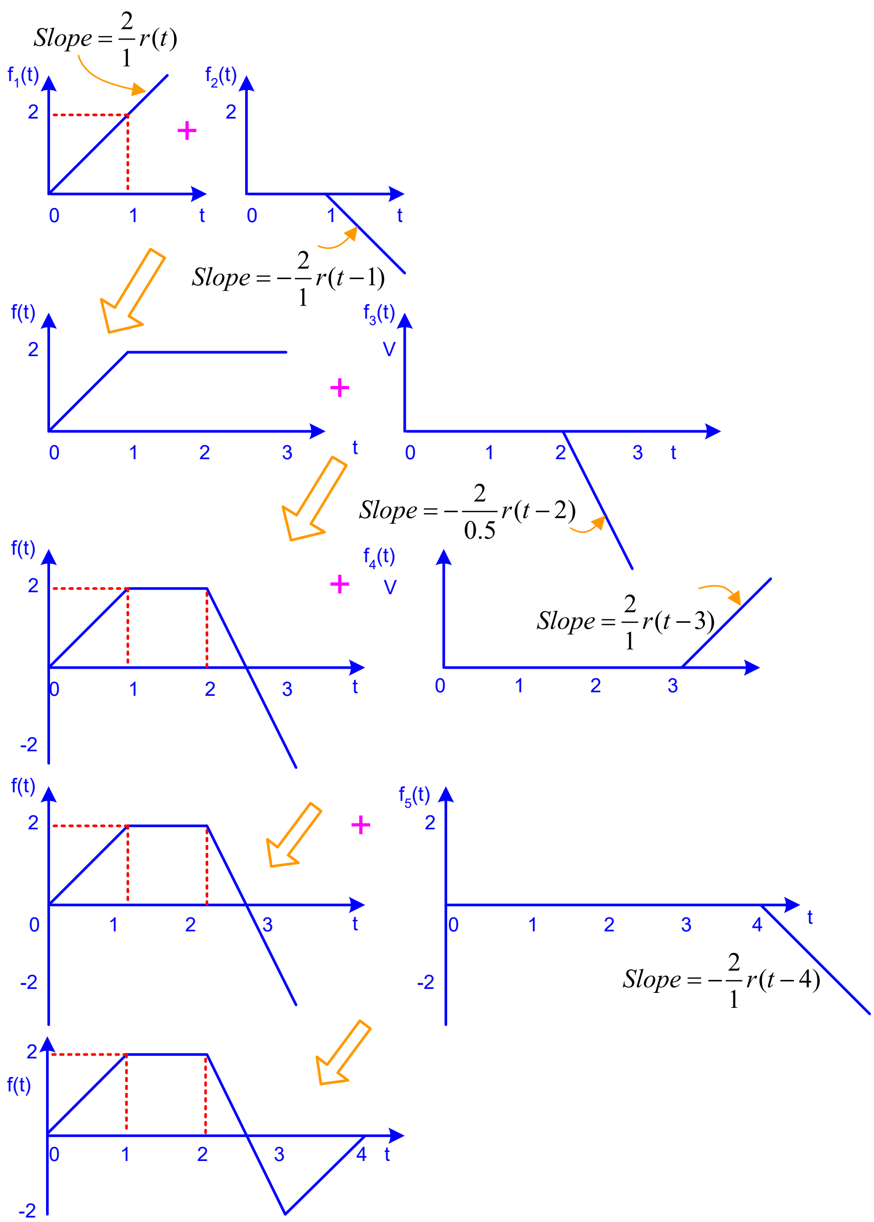

Figure 1.20: Trapezoid Wave

$$
\begin{aligned}
f(t) &= f_1(t) + f_2(t) + f_3(t) + f_4(t) + f_5(t) \\
&= 2r(t)u(t) - 2r(t-1)u(t-1) - 4r(t-2)u(t-2) + 6r(t-3)u(t-3) - 6r(t-4)u(t-4) \\
F(s) &= \frac{2}{s^2} - \frac{2}{s^2}e^{-2s} - \frac{4}{s^2}e^{-2s} + \frac{6}{s^2}e^{-3s} - \frac{6}{s^2}e^{-4s}
\end{aligned}
$$

Q 7) Find the Laplace transform of the waveform as shown in Figure 1.21.

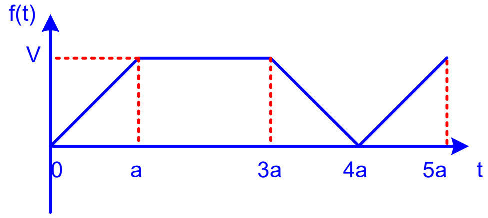

Figure 1.21: Trapezoid Wave

Solution:

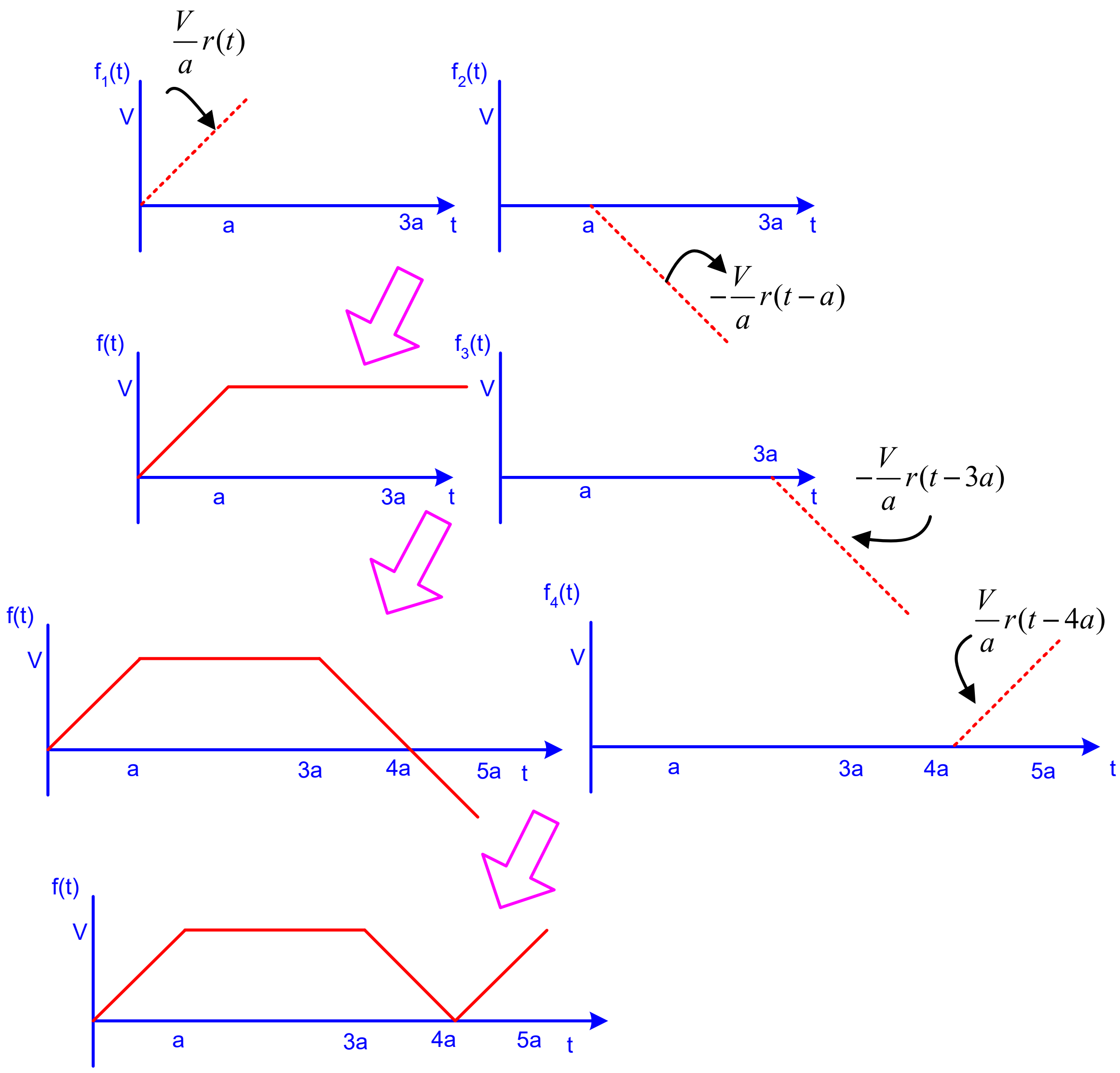

Figure 1.22: Trapezoid Wave

$$Slope = \frac{E}{t_0}$$

$$\begin{aligned}
f(t) &= f_1(t) + f_2(t) + f_3(t) + f_4(t) + f_5(t) \\
&= \frac{V}{a}r(t) - \frac{V}{a}r(t-a) - \frac{V}{a}r(t-3a) + \frac{V}{a}r(t-4a) \\
F(s) &= \frac{V}{a}[\frac{1}{s^2} - \frac{1}{s^2}e^{-as} - \frac{1}{s^2}e^{-3as} + \frac{1}{s^2}e^{-4as}]
\end{aligned}$$

Q 8) Find the Laplace transform of the waveform as shown in Figure 1.23.

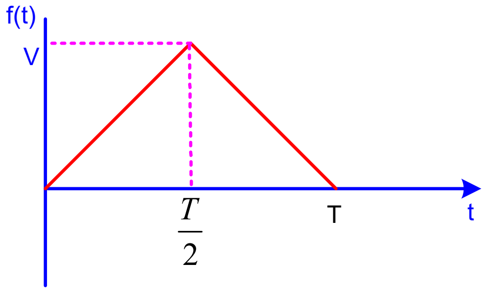

Figure 1.23: Trapezoid Wave

Solution:

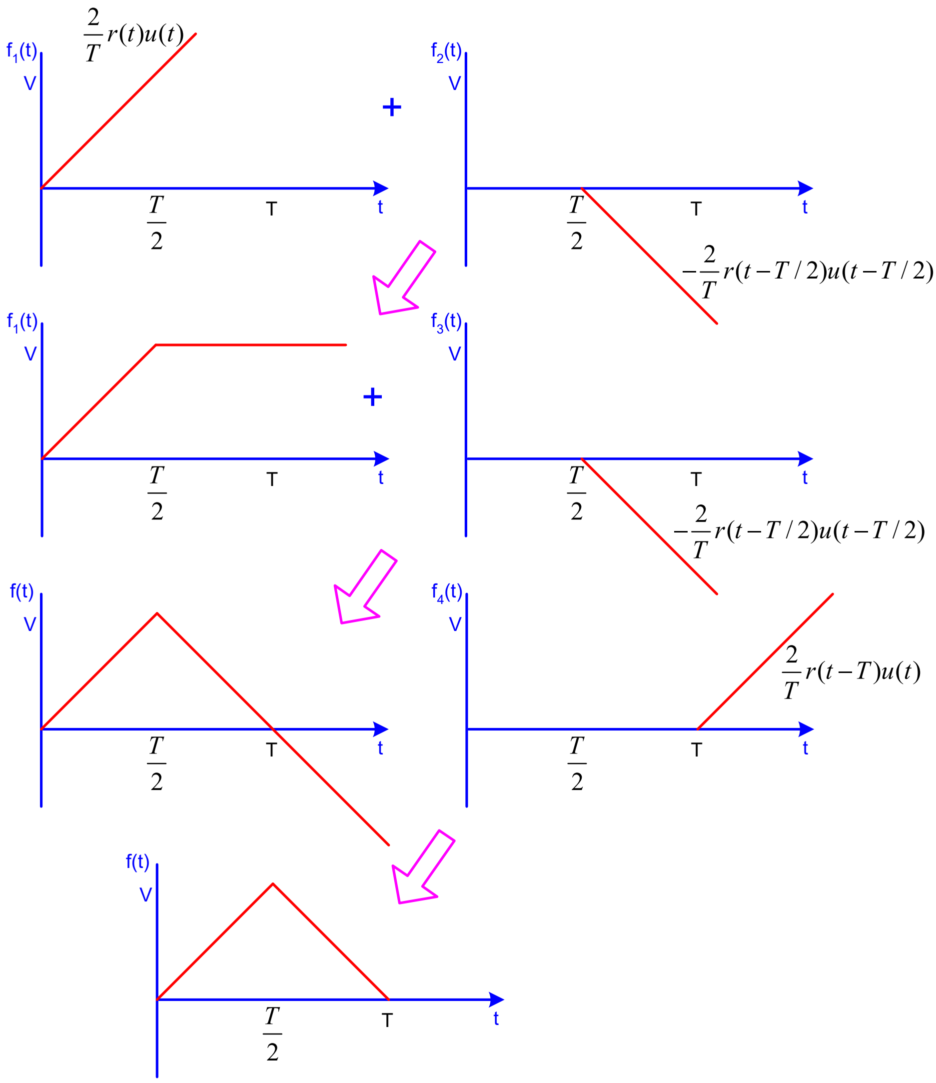

Figure 1.24: Trapezoid Wave

$$
\begin{aligned}
f(t) &= f_1(t) + f_2(t) + f_3(t) + f_4(t) \\
&= \frac{2}{T}r(t) - \frac{2}{T}r(t-T/2) - \frac{2}{T}r(t-T/2) + \frac{2}{T}r(t-T) \\
&= \frac{2}{T}r(t) - \frac{4}{T}r(t-T/2) + \frac{2}{T}r(t-T) \\
F(s) &= \frac{2}{T}[\frac{1}{s^2} - \frac{1}{s^2}e^{-T/2s} - \frac{1}{s^2}e^{-T/2s} + \frac{1}{s^2}e^{-Ts}]
\end{aligned}
$$

Q 9) Find the Laplace transform of the waveform as shown in Figure 1.25.

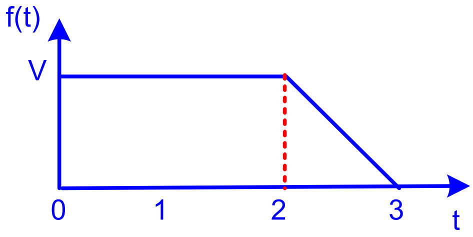

Figure 1.25: Trapezoid Wave

Solution:

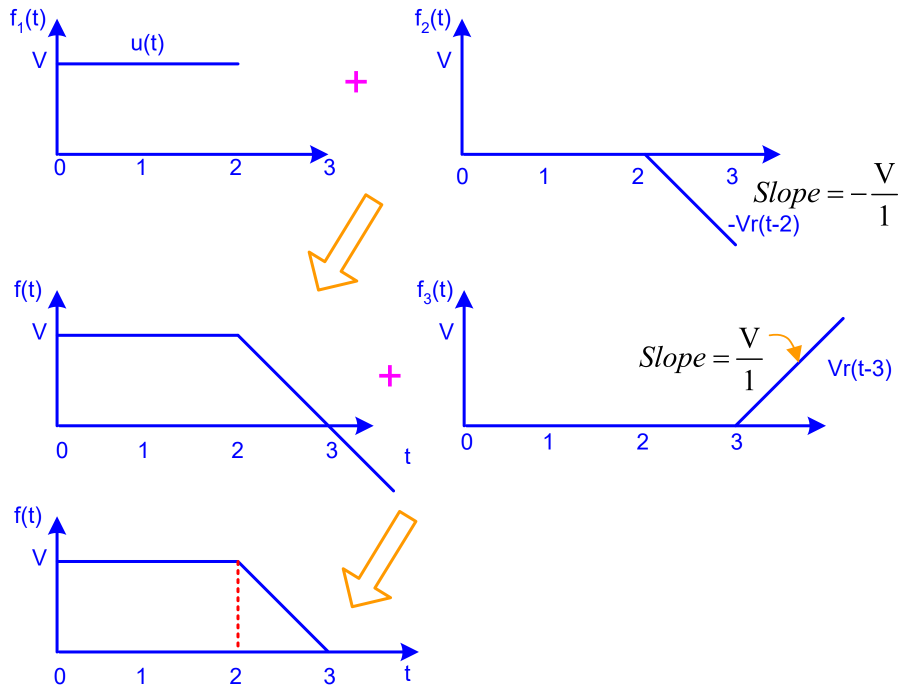

Figure 1.26: Trapezoid Wave

$$
\begin{aligned}
f(t) &= f_1(t) + f_2(t) + f_3(t) \\
&= u(t) + \frac{V}{1} r(t-2) + \frac{V}{1} r(t-3) \\
F(s) &= \frac{V}{s^2} - \frac{V}{s^2} e^{-2s} + \frac{V}{s^2} e^{-3s}
\end{aligned}
$$

Q 10) Find the Laplace transform of the waveform as shown in Figure 1.27.

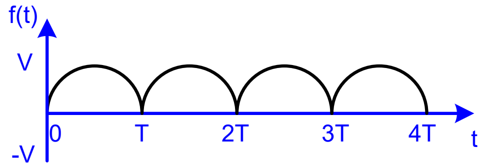

Figure 1.27: Square wave

Solution:

By considering one complete cycle from 0 to T

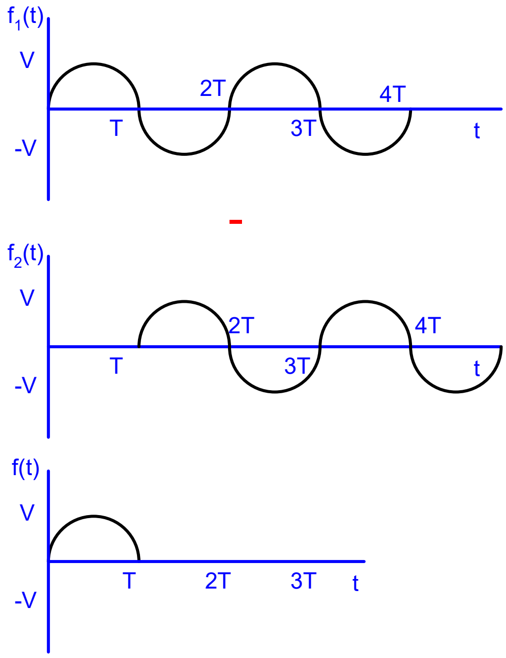

Figure 1.28: Square wave

Sinusoidal signal has a one cycle duration of 2T.

$$\omega = 2\pi f = 2\pi \frac{1}{2T} = \frac{\pi}{T}$$

$$L[sin\omega t] = \frac{\omega}{S^2 + \omega^2}$$

$$\begin{aligned} L[f_(t)] &= V sin\frac{\pi}{T}tu(t) + V sin\frac{\pi}{T}(t-T)u(t-T) \\ F_1(S) &= V\left[\frac{\frac{\pi}{T}}{S^2 + (\frac{\pi}{T})^2} + \frac{\frac{\pi}{T}}{S^2 + (\frac{\pi}{T})^2}e^{-TS}\right] \\ &= V\frac{\frac{\pi}{T}}{S^2 + (\frac{\pi}{T})^2}\left[1 + e^{-TS}\right] \end{aligned}$$

Laplace transform of the periodic waveform is

$$\begin{aligned} F(S) &= \frac{F_1(S)}{1 - e^{-TS}} \\ &= V\frac{\frac{\pi}{T}}{S^2 + (\frac{\pi}{T})^2}\left[1 + e^{-TS}\right]\frac{1}{(1 - e^{-TS})} \\ &= \frac{V}{TS^2(1 - e^{-TS})}\left[1 - e^{-TS} - TSe^{-TS}\right] \end{aligned}$$

Q 11) Find the Laplace transform of the waveform as shown in Figure 1.29.

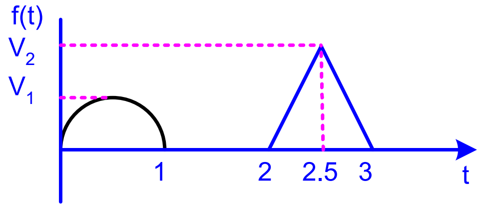

Figure 1.29: Trapezoid Wave

Solution:

$$\omega = 2\pi f = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi \ (For\ sine\ wave\ T = 2)$$

$$\begin{aligned}
f(t) &= f_1(t) + f_2(t) + f_3(t) \\
&= V_1 sin\pi t + V_1 sin\pi(t-1) + \frac{V_2}{0.5} r(t-2)u(t-2) - \frac{V_2}{0.25} r(t-2.5)u(t-2) + \frac{V_2}{0.5} r(t-3)u(t-3) \\
F(s) &= V_1 \frac{\pi}{s^2+\pi^2} + V_1 \frac{\pi}{s^2+\pi^2} e^{-s} + \frac{2V_2}{s^2} e^{-2s} - \frac{4V_2}{s^2} e^{-0.25s} + \frac{2V_2}{s^2} e^{-3s} \\
&= V_1 \frac{\pi}{s^2+\pi^2} \left[1 + e^{-s}\right] + \frac{2V_2}{s^2} \left[e^{-2s} - 2e^{-0.25s} + e^{-3s}\right]
\end{aligned}$$

Q 12 2009-JULY) Find the Laplace transform of the waveform as shown in Figure 1.30.

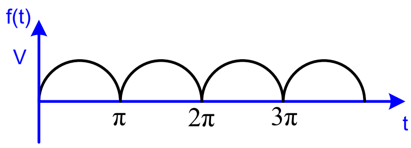

Figure 1.30: Full wave rectifier output

Solution:

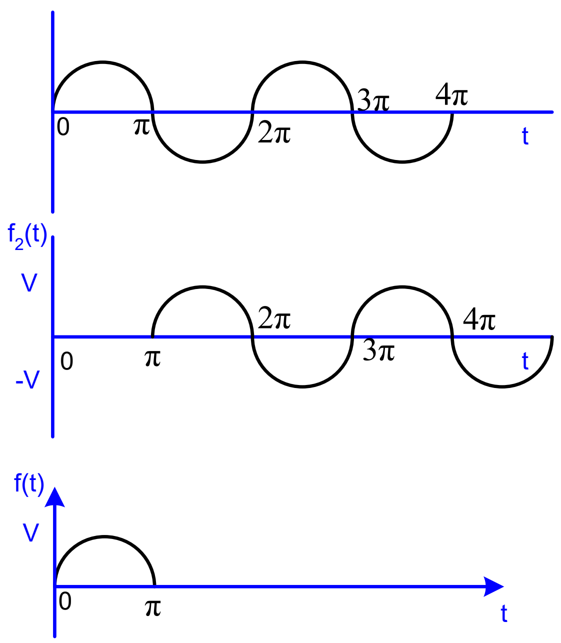

Figure 1.31: Full wave rectifier output

$$\omega = 2\pi f = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1 \ (For\ sine\ wave\ T = 2\pi)$$

$$\begin{aligned}
f(t) &= f_1(t) + f_2(t) \\
&= V sint + V sin(t-\pi) \\
F(s) &= V \frac{1}{s^2+1^2} + \frac{1}{s^2+1^2} e^{-\pi s} \\
&= V \frac{(1+e^{-\pi s})}{s^2+1} = \frac{V}{s^2+1}(1+e^{-\pi s})
\end{aligned}$$

For complete waveform

$$\begin{aligned}
F_3(s) &= \frac{F(s)}{1-e^{-st}} = \frac{F(s)}{1-e^{-\pi s}} \\
F_3(s) &= \frac{V}{s^2+1}(1+e^{-\pi s}) \frac{1}{1-e^{-\pi s}} \\
&= \frac{V}{s^2+1} \frac{(1+e^{-\pi s})}{(1-e^{-\pi s})}
\end{aligned}$$

Q 13) Find the Laplace transform of the waveform as shown in Figure 1.32.

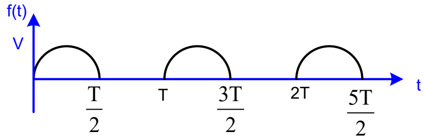

Figure 1.32: Full wave rectifier output

Solution:

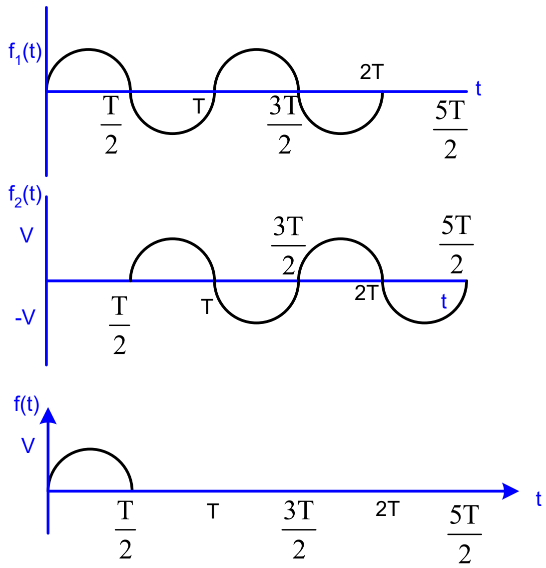

Figure 1.33: Full wave rectifier output

$$\omega = 2\pi f = \frac{2\pi}{T} = \frac{2\pi}{T} \; (For\ sine\ wave\ T = 2\pi)$$

$$\begin{aligned}
f(t) &= f_1(t) + f_2(t) \\
&= V sin\frac{2\pi}{T}t + V sin\frac{2\pi}{T}\left(t - \frac{T}{2}\right) \\
F(s) &= V\frac{\frac{2\pi}{T}}{s^2 + \left(\frac{2\pi}{T}\right)^2} + V\frac{\frac{2\pi}{T}}{s^2 + \left(\frac{2\pi}{T}\right)^2}e^{-\frac{T}{2}s} \\
&= V\frac{\frac{2\pi}{T}}{s^2 + \left(\frac{2\pi}{T}\right)^2}(1 + e^{-\frac{T}{2}s})
\end{aligned}$$

For complete waveform

$$\begin{aligned}
F_3(s) &= \frac{F(s)}{1 - e^{-st}} = \frac{F(s)}{1 - e^{-Ts}} \\
F_3(s) &= V\frac{\frac{2\pi}{T}}{s^2 + \left(\frac{2\pi}{T}\right)^2}\frac{(1 + e^{-\frac{T}{2}s})}{1 - e^{-Ts}}
\end{aligned}$$

Q 14) Find the Laplace transform of the waveform as shown in Figure 1.34.

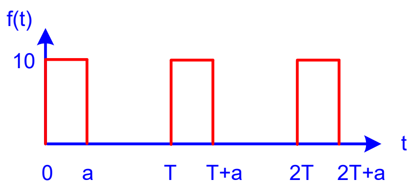

Figure 1.34: Full wave rectifier output

Solution:

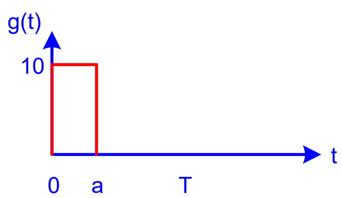

Figure 1.35: Full wave rectifier output

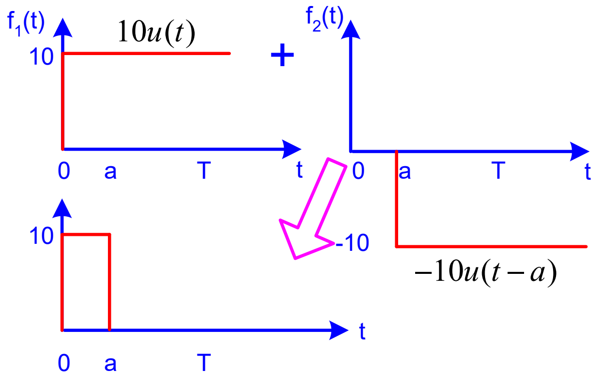

Figure 1.36: Full wave rectifier output

$$\begin{aligned}
g(t) &= f_1(t) + f_2(t) \\
&= 10u(t) - 10u(t - a) \\
G(s) &= \frac{10}{s} - \frac{10}{s}e^{-as} \\
&= \frac{10}{s}\left(1 - e^{-as}\right)
\end{aligned}$$

For complete waveform

$$F(s) = \frac{G(s)}{1 - e^{-st}} = \frac{10}{s}\left[\frac{1 - e^{-as}}{1 - e^{-Ts}}\right]$$

Q 15 2010-DEC) Find the Laplace transform of the waveform as shown in Figure 1.37.

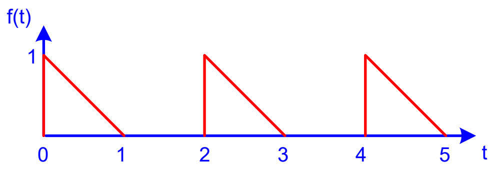

Figure 1.37

Solution:

By considering one cycle.

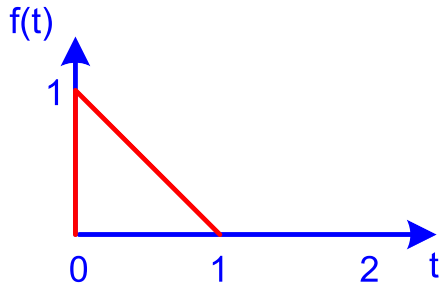

Figure 1.38

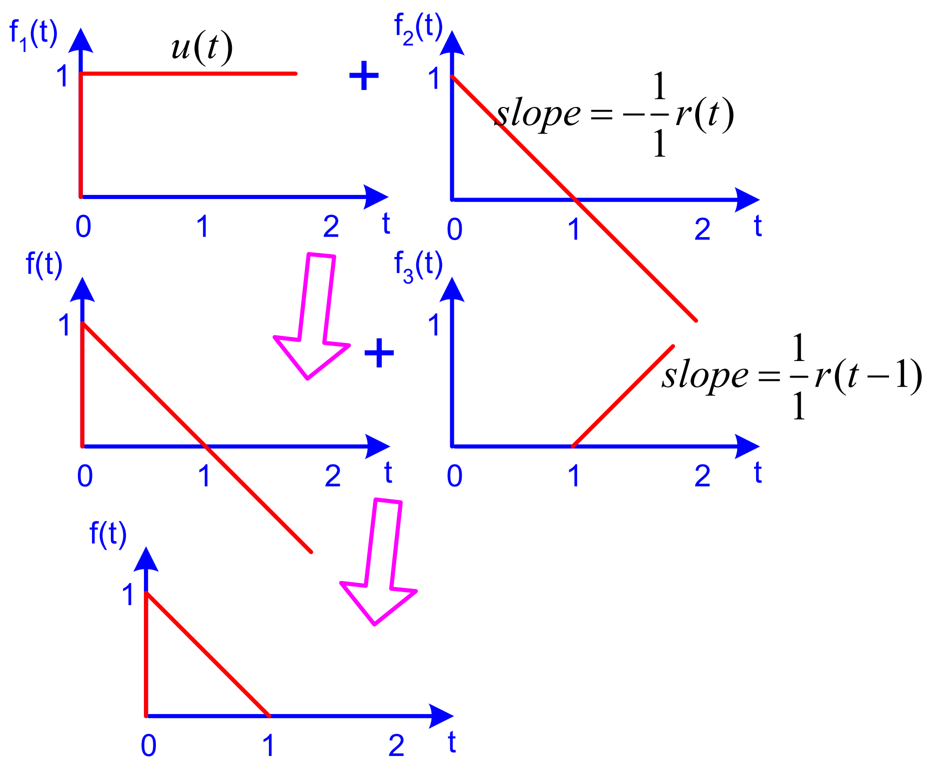

Figure 1.39

For complete waveform

$$
\begin{aligned}
f(t) &= f_1(t) + f_2(t) + f_2(t) \\
&= u(t) - tu(t) + tu(t-1)(\\
F(s) &= \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^2}e^{-s} = \frac{(s-1+e^{-s})}{s^2}
\end{aligned}
$$

$$
\begin{aligned}
F_3(s) &= \frac{F(s)}{1-e^{-st}} = \frac{F(s)}{1-e^{-2s}} \\
&= \frac{(s-1+e^{-s})}{s^2(1-e^{-2s})}
\end{aligned}
$$

Q 16 2009-JULY) Find the Laplace transform of the waveform as shown in Figure 1.40.

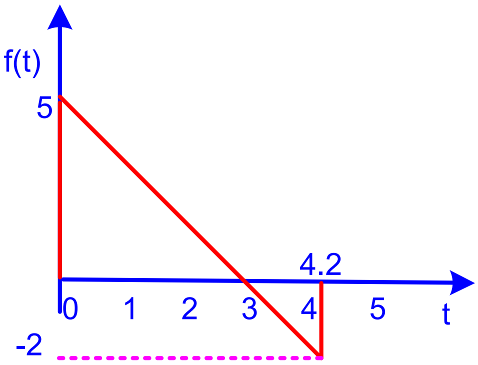

Figure 1.40

Solution:

By considering one cycle.

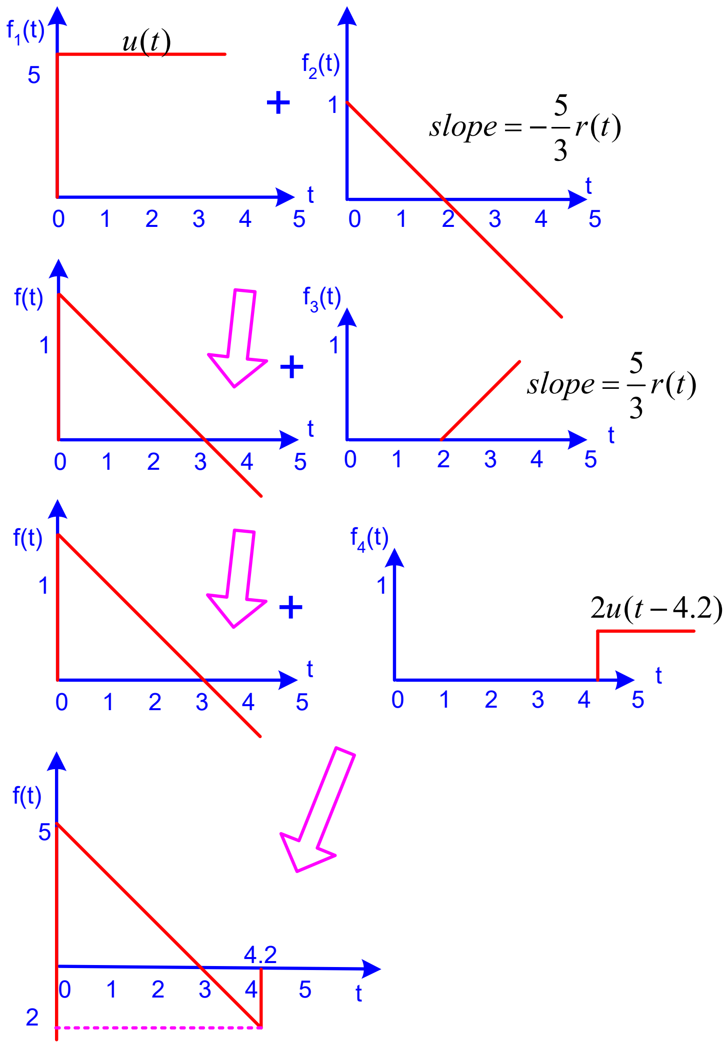

Figure 1.41

$$
\begin{aligned}
f(t) &= f_1(t) + f_2(t) + f_2(t) + f_2(t) \\
&= 5u(t) - \frac{5}{3}tu(t) + \frac{5}{3}tu(t-4.2) + 2u(t-4.2) \\
F(s) &= \frac{5}{s} - \frac{5}{3s^2} + \frac{5}{3s^2}e^{-4.2s} + \frac{2}{s}e^{-4.2s}
\end{aligned}
$$

1.1 Solutions

Q 1 2012-JUNE-7a) In the circuit shown in Figure 1.42 find an expression for $i(t)$ when the switch K is closed at t=0.

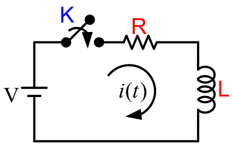

Figure 1.42: Example

Solution:

$$i(0^-) = 0 = i(0^+)$$

KVL for the given circuit is

$$V = Ri + L\frac{di}{dt}$$

Taking Laplace transform on both sides

$$\begin{aligned}\frac{V}{s} &= RI(s) + L[sI(s) - i(0^-)] \\ &= I(s)[R + Ls] \\ I(s) &= \frac{V}{s(R+Ls)} \\ &= \frac{V}{L}\left[\frac{1}{s(s+R/L)}\right]\end{aligned}$$

$$\frac{1}{s(s+R/L)} = \frac{A}{s} + \frac{B}{(s+R/L)}$$

$$\begin{aligned}A &= \frac{1}{s(s+R/L)} \times s|_{s=0} \\ &= L/R\end{aligned}$$

$$\begin{aligned}B &= \frac{1}{s(s+R/L)} \times (s+R/L)|_{s=-R/L} \\ &= -L/R\end{aligned}$$

$$\begin{aligned}I(s) &= \frac{V}{L}\left[\frac{L/R}{s} - \frac{L/R}{s+R/L}\right] \\ i(t) &= \frac{V}{L}\left[\frac{L}{R} - \frac{L}{R}e^{-\frac{R}{L}t}\right] \\ &= \frac{V}{R}\left[1 - e^{-\frac{R}{L}t}\right]\end{aligned}$$

Q 2) In the circuit shown in Figure 1.43 find an expression for $i(t)$ when the switch K is closed at t=0. Assume that there is no initial charge on the capacitor.

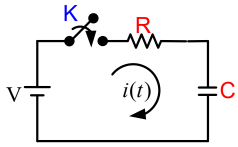

Figure 1.43: Example

Solution: Assuming that there is no initial charge on the capacitor then

$$q(0^-) = 0 = q(0^+) \tag{1.1}$$

KVL for the given circuit is

$$\begin{aligned}V &= Ri + \frac{1}{c}\int idt \\ \frac{V}{s} &= RI(s) + \frac{1}{c}\left[\frac{I(s)}{s} + \frac{q(0^-)}{s}\right] \\ &= I(s)\left[R + \frac{1}{Cs}\right] \\ I(s) &= \frac{V}{s\left[R + \frac{1}{Cs}\right]} \\ &= \frac{V}{R\left[s + \frac{1}{RC}\right]} = \frac{V}{R}\frac{1}{\left[s + \frac{1}{RC}\right]} \\ i(t) &= \frac{V}{R}e^{-\frac{t}{RC}}\end{aligned}$$

Q 3 2014-JAN-7a) In the circuit shown in Figure 1.44 the battery voltage 10 V is applied for a steady state period with switch K is open. Obtain the expression for the current after closing the switch K. Use laplace transform.

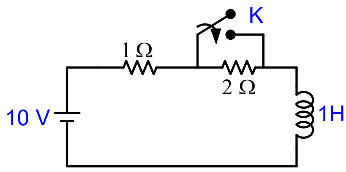

Figure 1.44: Example

Solution:

$$i(0^-) = \frac{10}{3} = i(0^+) \tag{1.2}$$

KVL for the given circuit is

$$\begin{aligned}10 &= Ri + L\frac{di}{dt} \\ \frac{10}{s} &= 1I(s) + 1[sI(s) - Li(0^+)] \\ \frac{10}{s} &= I(s)(1+s) - \frac{10}{3} \\ I(s)(1+s) &= \frac{10}{s} + \frac{10}{3} \\ I(s) &= \frac{10}{s(1+s)} + \frac{10}{3(1+s)}\end{aligned}$$

$$\frac{10}{s(1+s)} = \frac{A}{s} + \frac{B}{(1+s)}$$

$$\begin{aligned} A &= \frac{10}{s(1+s)} \times s|_{s=0} \\ &= 10 \end{aligned}$$

$$\begin{aligned} B &= \frac{10}{s(1+s)} \times (1+s)|_{s=-1} \\ &= -10 \end{aligned}$$

$$\begin{aligned} I(s) &= \frac{10}{s} - \frac{10}{(1+s)} + \frac{10}{3(1+s)} \\ I(s) &= \frac{10}{s} - \frac{6.67}{(1+s)} \\ i(t) &= [10 - 6.67e^{-t}]A \end{aligned}$$

Q 4 2013-JUNE-7a) Using Laplace transform obtain expression for the circuit shown in Figure 1.45. Assume zero initial conditions.

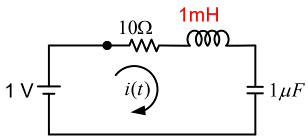

Figure 1.45: Example

Solution:

$$i(0^-) = i(0^+) = 0$$

$$V = 10i + L\frac{di}{dt} + \frac{1}{C}\int idt$$

Taking the Laplace transform on both sides

$$\begin{aligned} \frac{1}{s} &= 10I(s) + L[sI(s) - i(0^+)] \\ &\quad + \frac{1}{C}\left[\frac{I(s)}{s} + \frac{q(0^-)}{s}\right] \\ \frac{1}{s} &= 10I(s) + LsI(s) + \frac{1}{C}\frac{1}{s} \\ &= I(s)[10 + 10^{-3}s + \frac{10^6}{s}] \\ 11 &= I(s)[10s + 10^{-3}s^2 + 10^6] \\ 1 &= I(s)[10^{-3}s^2 + 10s + +10^6] \\ I(s) &= \frac{1}{10^{-3}s^2 + 10s + +10^6} \end{aligned}$$

Q 5 2013-JUNE-7b) For the critically related network of the circuit shown in Figure 1.46 obtain an expression for $i(t)$. Use Laplace transform.

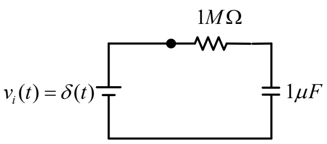

Figure 1.46: Example

Solution:

$$i(0^-) = i(0^+) = 0$$

$$v_i = Ri + \frac{1}{C}\int idt$$

$$\delta(t) = Ri + \frac{1}{C}\int idt$$

Taking the Laplace transform on both sides

$$\begin{aligned} V_i(s) &= RI(s) + \frac{1}{C}\left[\frac{I(s)}{s} + i\frac{(0^-)}{s}\right] \\ 1 &= RI(s) + \frac{1}{C}\frac{I(s)}{s} \\ 1 &= I(s)\left[\frac{RCs+1}{Cs}\right] \\ I(s) &= \left[\frac{Cs}{RCs+1}\right] = \left[\frac{10^{-6}s}{10^{-6}10^6 s + 1}\right] \\ I(s) &= 10^{-6}\frac{s}{s+1} \end{aligned}$$

$$\begin{aligned} &= 10^{-6}\frac{(s+1)-1}{s+1} \\ &= 10^{-6}\left[\frac{(s+1)}{s+1} - \frac{1}{s+1}\right] \\ &= 10^{-6}\left[1 - \frac{1}{s+1}\right] \\ i(t) &= 10^{-6}[\delta(t) - e^{-t}] \end{aligned}$$

Q 6 In the circuit shown in Figure 1.47 if the capacitor is initially charged to 1 V, find an expression for $i(t)$, when the switch K is closed at $t = 0$.Use Laplace transform

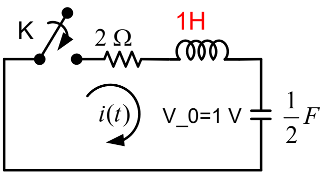

Figure 1.47: Example

Solution:

When the switch K is closed
$i(0^-) = 0A$ $\frac{q(0^-)}{s} = 1$

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int idt = 0$$

Taking the Laplace transform on both sides

$$2I(s) + L[sI(s) - i(0^+)] + \frac{1}{C}\left[\frac{I(s)}{s} + \frac{q(0^-)}{s}\right] = 0$$

$$2I(s) + LsI(s) + 2\left[\frac{I(s)}{s} - \frac{1}{s}\right] = 0$$

$$2I(s) + sI(s) + 2\frac{I(s)}{s} = \frac{2}{s}$$

$$I(s)\left[2 + s + \frac{2}{s}\right] = \frac{2}{s}$$

$$I(s)\left[\frac{2s + s^2 + 2}{s}\right] = 2$$

$$\begin{aligned} I(s) &= 2\tfrac{1}{s^2+2s+2} \\ I(s) &= 2\tfrac{1}{(s+1)^2+1} \\ i(t) &= 2e^{-t}sint \end{aligned}$$

Q 7 In the circuit shown in Figure 1.48 the the switch K is closed and steady state is reached. At $t = 0$ switch K is opened. Find the expression for current $i(t)$ in the inductor using Laplace transform.

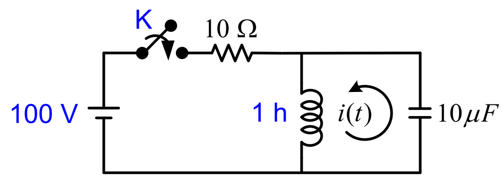

Figure 1.48: Example

Solution:

When the switch K is closed $i(0^-) = \frac{100}{10} = 10A = i(0^+)$ $V_c(0^-) = 0 = V_c(0^+)$ $q(0^-) = 0 = q(0^+)$
When the switch K is opened

$$L\frac{di}{dt} + \frac{1}{C}\int idt = 0$$

Taking the Laplace transform on both sides

$$L[sI(s) - i(0^+)] + \frac{1}{C}\left[\frac{I(s)}{s} + \frac{q(0^-)}{s}\right] = 0$$

$$LsI(s) - Li(0^+) + \frac{I(s)}{Cs} = 0$$

$$1I(s) - 10 + \frac{I(s)}{10 \times 10^{-6}} = 0$$

$$I(s)\left[s + \frac{10^5}{s}\right] = 10$$

$$\begin{aligned} I(s) &= \tfrac{10s}{s^2+10^5} \\ I(s) &= \tfrac{10s}{s^2+(10^{5/2})^2} \\ i(t) &= 10cos(10^{5/2})t \end{aligned}$$

Q 1 2017-JAN-7a) Using Laplace method obtain the expression for $i(t)$. The capacitor is zero initially. Also obtain the expression for the capacitor voltage in S domain for the circuit shown in Figure 1.49.

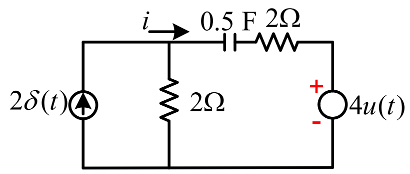

Figure 1.49: Example

Solution:

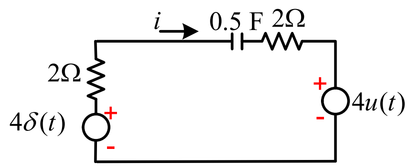

Figure 1.50: Example

$$\begin{aligned} 0 &= 4i(t) + \frac{1}{0.5}\int_0^\infty i(t)dt + 4u(t) - 4\delta(t) \\ &= 4I(s) + 2\frac{I(s)}{s} + 4\frac{1}{s} - 4 \\ &= I(s)\left[4 + \frac{2}{s}\right] + \left[\frac{4-4s}{s}\right] \\ &= I(s)\left[\frac{4s+2}{s}\right] - \left[\frac{4s-4}{s}\right] \\ I(s)\left[\frac{4s+2}{s}\right] &= \left[\frac{4s-4}{s}\right] \\ I(s) &= \frac{4s-4}{4s+2} = \frac{4s}{4s+2}\frac{-4}{4s+2} \\ I(s) &= \frac{4s}{4s+2} - \frac{4}{4(s+0.5)} \\ &= \frac{4s}{4s+2} - \frac{1}{s+0.5} \end{aligned}$$

Q 1 2014-JAN-7b) In the circuit shown in Figure 1.44 solve for $i_L(t)$ using Laplace transformation.

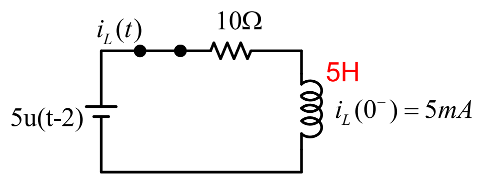

Figure 1.51: Example

Solution:

$$i(0^-) = 5mA = i(0^+) \tag{1.3}$$

KVL for the given circuit is

$$\begin{aligned}
5u(t-2) &= 10i_L(t) + 5\frac{di}{dt} \\
\frac{5}{s}e^{-2s} &= 10I_L(s) + 5[sI_L(s) - i_L(0^+)] \\
\frac{1}{s}e^{-2s} &= 2I_L(s) + [sI_L(s) - i_L(0^+)] \\
\frac{1}{s}e^{-2s} &= I_L(s)(2+s) - 5\times 10^{-3} \\
I_L(s)(2+s) &= \frac{1}{s}e^{-2s} + 5\times 10^{-3} \\
I_L(s) &= \frac{1}{s(s+2)}e^{-2s} + \frac{5\times 10^{-3}}{s+2}
\end{aligned}$$

$$\frac{1}{s(s+2)} = \frac{A}{s} + \frac{B}{(s+2)}$$

$$\begin{aligned}
A &= \frac{10}{s(s+2)} \times s|_{s=0} \\
&= \frac{1}{2} \\
B &= \frac{1}{s(s+2)} \times (s+2)|_{s=-2} \\
&= -\frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
I(s) &= \frac{1}{2}\left[\frac{1}{s} - \frac{1}{(s+2)}\right]e^{-2s} + \frac{5\times 10^{-3}}{s+2} \\
I(s) &= \frac{1}{2}\left[\frac{e^{-2s}}{s} - \frac{e^{-2s}}{(s+2)}\right] + \frac{5\times 10^{-3}}{s+2} \\
i(t) &= \frac{1}{2}\left[u(t-2) - e^{-2(t-2)}u(t-2)\right] \\
&\quad + 5\times 10^{-3}e^{-2t}u(t)
\end{aligned}$$

Q 1 2012-JUNE-7b) Find the laplace transform of the given function $f(t) = 5 + 4e^{-2t}$.

Solution:

$$\begin{aligned}
f(t) &= 5 + 4e^{-2t} \\
F(s) &= \int_0^\infty 5e^{-st}dt + \int_0^\infty 4e^{-2t}e^{-st}dt \\
&= 5\left[\frac{e^{-st}}{s} + 0\right]_0^\infty + 4\int_0^\infty e^{-(s+2)t}dt \\
&= 5\left[\frac{e^{-st}}{s} + 0\right]_0^\infty + 4\left[\frac{e^{-(s+2)t}}{-(s+2)} + 0\right]_0^\infty \\
&= \frac{5}{s} + \frac{4}{s+2} = \frac{5s+10+4s}{s(s+2)} \\
&= \frac{9s+10}{s(s+2)}
\end{aligned}$$

Q 2012-DEC-7b) For the circuit shown in Figure 1.52 was in steady state before t=0. The switch opened at t=0. Find $i(t)$ $i(t) > 0$ using Laplace transform.

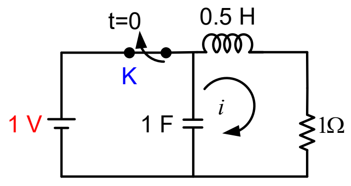

Figure 1.52: Example

Solution:

When the steady state is reached the circuit is as shown in Figure 1.53, capacitor is fully charged with voltage $v_c(0^-) = 1V$ and inductor current is

$$I_L(0^-) = \frac{1}{1} = 1V = i(0^+)$$

$$v_c(0^-) = v_c(0^+) = 1V$$

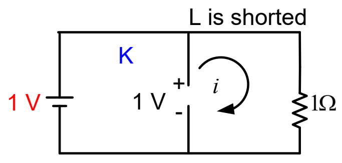

Figure 1.53: Example

When the switch is opened the circuit is as shown in Figure 1.54

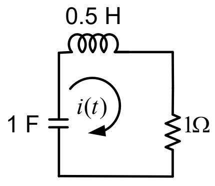

Figure 1.54: Example

$$\begin{aligned}
0 &= 0.5\frac{di(t)}{dt} + i(t) + \frac{1}{C}\int_{-\infty}^{t} i(t)dt \\
0 &= 0.5\frac{di(t)}{dt} + i(t) + \frac{1}{C}\int_{-\infty}^{0} i(t)dt + \frac{1}{C}\int_0^t i(t)dt \\
0 &= 0.5\frac{di(t)}{dt} + i(t) - v_c(0^-) + \frac{1}{C}\int_0^t i(t)dt \\
1 &= 0.5\frac{di(t)}{dt} + i(t) + \frac{1}{C}\int_0^t i(t)dt
\end{aligned}$$

Taking the Laplace transform on both sides

$$\begin{aligned}
\frac{1}{s} &= 0.5\left[sI(s)-i(0^-)\right]+I(s)+\frac{I(s)}{s}\\
\frac{1}{s}+0.5 &= I(s)\left[0.5s+1+\frac{1}{s}\right]\\
\frac{1+0.5s}{s} &= I(s)\left[\frac{0.5s^2+s+1}{s}\right]\\
I(s) &= \frac{1+0.5s}{0.5s^2+s+1}=\frac{0.5[s+2]}{0.5[2s^2+2s+2]}\\
&= \frac{s+2}{2s^2+2s+2}=\frac{s+2}{(s+1)^2+1}\\
&= \frac{s+1+1}{(s+1)^2+1^2}\\
&= \frac{s+1}{(s+1)^2+1^2}+\frac{1}{(s+1)^2+1^2}\\
i(t) &= e^{-t}cost+e^{-t}sint
\end{aligned}$$

$$\begin{aligned}
L[sin\omega t] &= \frac{\omega}{S^2+\omega^2} \quad L[cos\omega t]=\frac{S}{S^2+\omega^2}\\
L[te^{-at}] &= F(S+a)
\end{aligned}$$

Q 1 2011-JUNE-7a) Find the current $i(t)$ when switch K is opened at t=0 with the circuit having reached steady state before the switching in Fig 1.55. Find current at t=0.5 sec

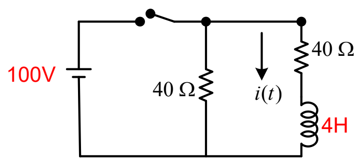

Figure 1.55: Example

Solution:

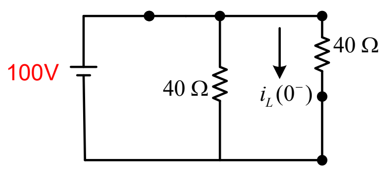

Figure 1.56: Example

At $t=0^-$ inductor acts as a short circuit which is as shown in Figure 1.56and the current $i_l(t)=0^-$ at $t=0^-$ is

$$i_l(0^-)=\frac{100}{40}=2.5A=i_l(0^+)$$

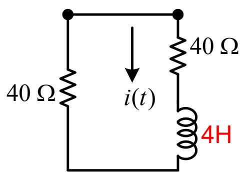

Figure 1.57: Example

Applying KVL for the circuit as shown in Figure is 1.57

$$\begin{aligned}
0 &= 40i(t)+L\frac{di}{dt}+40i(t)=80i(t)+4\frac{di}{dt}\\
0 &= 4I\left[sI(s)-i(0)\right]+80I(s)\\
0 &= 4sI(s)-10+80I(s)=I(s)[4s+80]\\
10 &= I(s)[4s+80]\\
I(s) &= \frac{10}{4s+80}=\frac{10}{4(s+20)}=\frac{2.5}{s+20}\\
i(t) &= 2.5e^{-20t}
\end{aligned}$$

The current at t=0.5 sec is

$$i(t)=2.5e^{-20\times 0.5}=1.135\times 10^{-4}A$$

Q 1 2011-JUNE-7b) Find the current $i(t)$ assuming zero initial conditions when switch K is closed at t=0. The excitation $v(t)$ is a pulse of magnitude of 10 V and duration of 2 sec. Refer Figure 1.58.

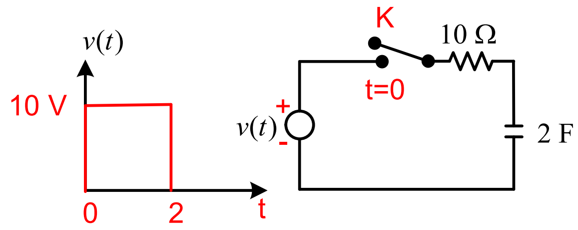

Figure 1.58: Example

Solution:

The laplace transform of the input pulse is

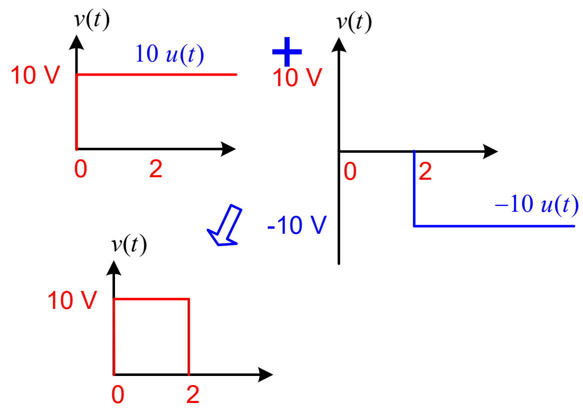

Figure 1.59: Example

$$\begin{aligned}
v(t) &= 10u(t)-10u(t-2)\\
V(s) &= \frac{10}{s}-\frac{10}{s}e^{-2s}\\
&= \frac{10}{s}\left[1-e^{-2s}\right]
\end{aligned}$$

Applying KVL for the circuit shown in Figure1.58

$$\begin{aligned}
v(t) &= Ri + \frac{1}{C}\int idt \\
V(s) &= 10I(s) + \frac{1}{Cs}I(s) = I(s)\left(10 + \frac{1}{2s}\right) \\
&= I(s)\left(\frac{20s+1}{2s}\right) \\
I(s) &= \frac{V(s)}{\frac{20s+1}{2s}} = \frac{2sV(s)}{20s+1} = \frac{2sV(s)}{20(s+0.05)} \\
&= \frac{0.1s}{s+0.05} \\
I(s) &= \frac{0.1s}{s+0.05}\left[\frac{10}{s}\left[1-e^{-2s}\right]\right] \\
&= \frac{1}{s+0.05}\left[1-e^{-2s}\right] = \frac{1}{s+0.05} - \frac{e^{-2s}}{s+0.05} \\
i(t) &= e^{-0.005t}u(t) - e^{-0.005(t-2)}u(t-2)
\end{aligned}$$

Q 1 2011-DEC-7c) State and prove initial value theorem

Solution:

Initial value theorem is used to find the initial value of $x(t)$ at $t = 0$ i.e., $x(0)$ directly from the Laplace transform $X(s)$.

It states that if $x(t)$ is a causal signal then

$$x(0) = \lim_{s\to\infty} sX(s)$$

Proof

$$L\left[\frac{dx(t)}{dt}\right] = sX(s) - x(0^-)$$

Taking left hand side of the term and limit as $s \to \infty$

$$L\left[\frac{dx(t)}{dt}\right] = \lim_{t\to\infty}\int_0^\infty e^{-st}\left[\frac{dx(t)}{dt}\right]dt$$

As $s \to \infty$ then $e^{-st} = 0$

$$\begin{aligned}
0 &= sX(s) - x(0) \\
x(0) &= sX(s)
\end{aligned}$$

Q 1 2011-DEC-7c) State and prove final value theorem

Solution:

Final value theorem is used to find the initial value of $x(t)$ at $t = \infty$ i.e., $x(\infty)$ directly from the Laplace transform $X(s)$.

It states that if $x(t)$ is a causal signal then

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

Proof

The Laplace transform of $\frac{dx(t)}{dt}$ is

$$L\left[\frac{dx(t)}{dt}\right] = sX(s) - x(0^-)$$

Taking left hand side of the term and limit as $s \to \infty$

$$\begin{aligned}
\lim_{s\to\infty} L\left[\frac{dx(t)}{dt}\right] &= \lim_{s\to\infty}\int_0^\infty \frac{dx(t)}{dt}e^{-st}dt \\
&= \int_0^\infty \frac{dx(t)}{dt}\left[\lim_{s\to 0} e^{-st}\right]dt \\
&= \int_0^\infty \frac{dx(t)}{dt}dt \\
&= [x(t)]_0^\infty = x(\infty) - x(0) \\
x(\infty) - x(0) &= \lim_{s\to\infty}[sX(s) - x(0)] \\
x(\infty) &= \lim_{s\to\infty}[sX(s)]
\end{aligned}$$
