Linear Block Codes[1]

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1. Generator and Parity check Matrices

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2. Encoding circuits

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3. Syndrome and Error Detection

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10. The (24, 12) Golay code
11. Product codes and Interleaved codes

Introduction to Linear Block Codes

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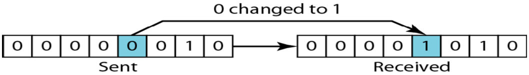

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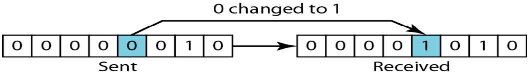

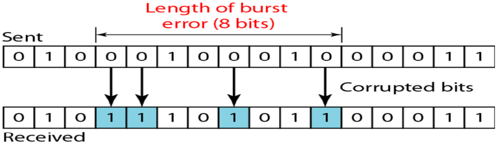

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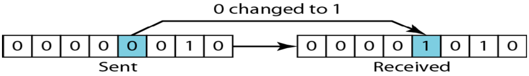

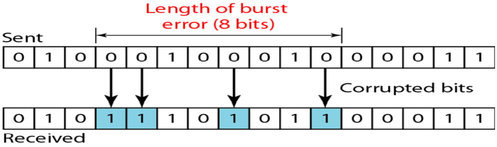

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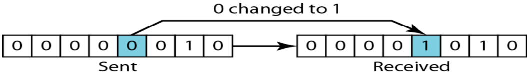

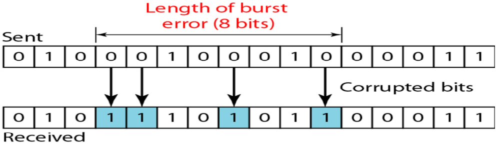

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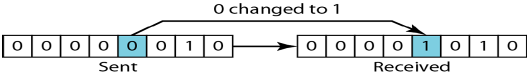

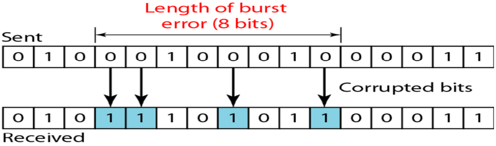

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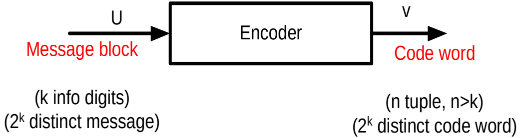

Figure: The encoder

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Message	Codewords
(0 0 0 0)	(0 0 0 0 0 0 0)
(1 0 0 0)	(1 1 0 1 0 0 0)
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(1 1 0 0)	(1 0 1 1 1 0 0)
(0 0 1 0)	(1 1 1 0 0 1 0)
(1 0 1 0)	(0 0 1 1 0 1 0)
(0 1 1 0)	(1 0 0 0 1 1 0)
(1 1 1 0)	(0 1 0 1 1 1 0)
(0 0 0 1)	(1 0 1 0 0 0 1)
(1 0 0 1)	(0 1 1 1 0 0 1)
(0 1 0 1)	(1 1 0 0 1 0 1)
(1 1 0 1)	(0 0 0 1 1 0 1)
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$$G = \begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_{k-1} \end{bmatrix} = \begin{bmatrix} g_{00} & g_{01} & g_{02} & \cdots & g_{0,n-1} \\ g_{10} & g_{11} & g_{12} & \cdots & g_{1,n-1} \\ \vdots & & & & \\ g_{k-1,0} & g_{k-1,1} & g_{k-1,2} & \cdots & g_{k-1,n-1} \end{bmatrix} \tag{2}$$

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where $g_i = (g_{i0}, g_{i1}, \ldots g_{in-1})0$ or 1 for $0 \leq i < k$

If $u = (u_0, u_1, \ldots, u_{k-1})$ is the message to be encoded, the corresponding code word

$$v = U.G = \begin{pmatrix} u_0, & u_1, \ldots, & u_{k-1} \end{pmatrix} . \begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_{k-1} \end{bmatrix} \tag{3}$$

$v = u_0 g_0 + u_1 g_1 + \ldots u_{k-1} g_{k-1}$

- Because the rows of G generate the (n, k) linear code C, the matrix G is called a generator matrix for C
- Note that any k linearly independent code words of an (n, k) linear code can be used to form a generator matrix for the code
- It follows from (3.3) that an (n, k) linear code is completely specified by the k rows of a generator matrix G

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- =1.(1 1 0 1 0 0 0)+ 1.(0 1 1 0 1 0 0)+ 0.(1 1 1 0 0 1 0)+1. (1 0 1 0 0 0 1)
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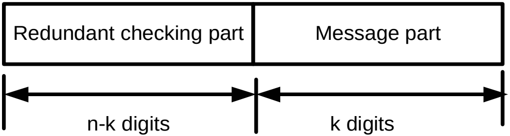

Figure: Systematic format of a codeword

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$$= \left[\begin{array}{cccc|ccccc} \multicolumn{4}{c}{\overbrace{\hphantom{p_{00} \quad p_{01} \quad \cdots \quad p_{0,n-k-1}}}^{P \quad Matrix}} & \multicolumn{5}{c}{\overbrace{\hphantom{1 \quad 0 \quad 0 \quad \ldots \quad 0}}^{kxk \quad Identity \quad Matrix}} \\ p_{00} & p_{01} & \cdots & p_{0,n-k-1} & 1 & 0 & 0 & \ldots & 0 \\ p_{10} & p_{11} & \cdots & p_{1,n-k-1} & 0 & 1 & 0 & \ldots & 0 \\ p_{20} & p_{21} & \cdots & p_{2,n-k-1} & 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & & & & & & \\ p_{k-1,0} & p_{k-1,1} & \cdots & p_{k-1,n-k-1} & 0 & 0 & 0 & \ldots & 1 \end{array} \right] \tag{4}$$

where $p_{ij} = 0$ or 1

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$$v_j = u_0 p_{0j} + u_1 p_{1j} + \ldots, + u_{k-1} p_{k-1,j} \quad for \;\; 0 \le j < n-k \quad (6b) \quad (6)$$

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- Equation (6b) shown that the leftmost n-k redundant digits are linear sums of the information digits.
- The n-k equations given by (6b) are called parity-check equations of the code.

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- An (n, k) linear code is completely specified by its parity check

$$v.H^T = (v_0, v_1, \dots v_{n-k-1}, u_0, u_1, \dots u_{k-1}).H^T = 0$$

$$H^T = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \\ p_{00} & p_{01} & p_{02} & \cdots & p_{0,n-k-1} \\ p_{10} & p_{11} & p_{12} & \cdots & p_{1,n-k-1} \\ \vdots & & & & \\ p_{k-1,0} & p_{k-1,1} & p_{k-1,2} & \cdots & p_{k-1,n-k-1} \end{bmatrix}$$

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$v_0(1) + v_1(1) + v_{n-1}(1) + u_0(p_{00}\ p_{01}\ \ldots\ p_{0,n-k-1})+$
$+u_1(p_{10}\ p_{11}\ p_{12} \ldots p_{1,n-k-1}) + u_2(p_{20}\ p_{21}\ p_{22} \ldots p_{2,n-k-1})$
$\ldots + u_{k-1}(p_{k-1,0}\ p_{k-1,1} \ldots p_{k-1,2}\ p_{k-1,n-k-1}) = 0$
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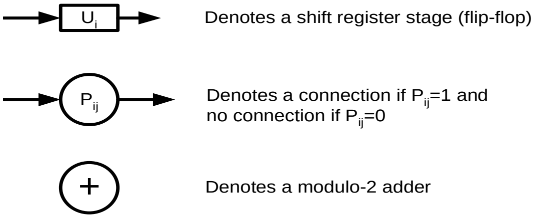

$v_{n-k+i} = u_i \qquad for \quad 0 \leq i < k$

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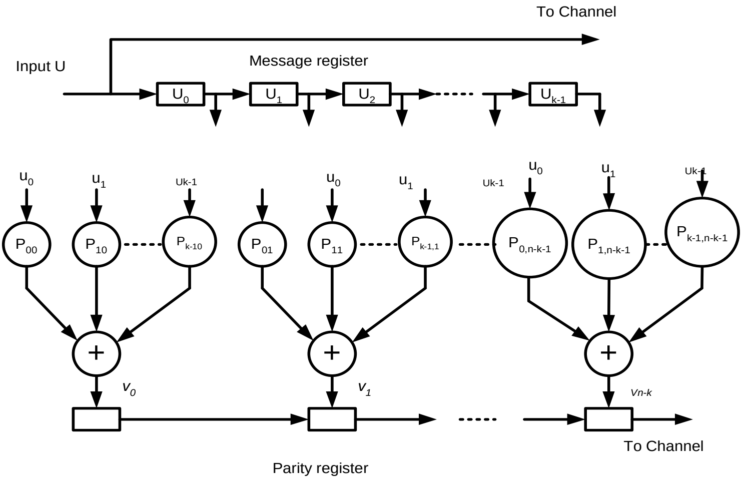

Figure: The encoding circuit for a liner system (n,k) code

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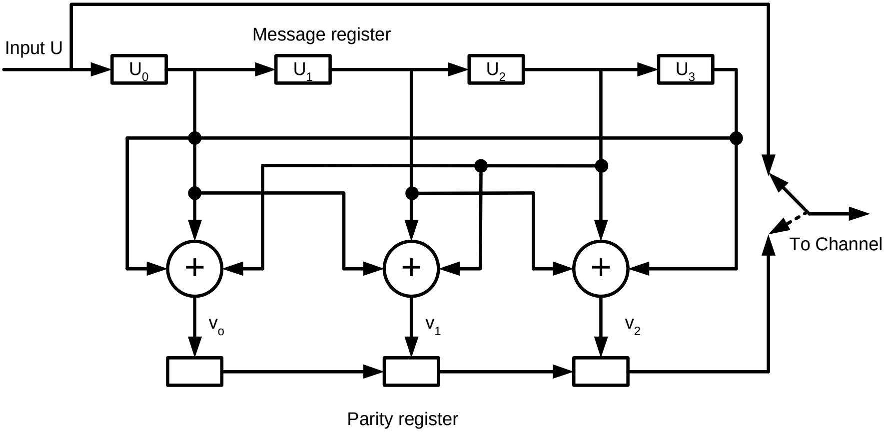

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Syndrome and Error Detection

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$$e = r + v = (e_0, e_1, \ldots, e_{n-1}) \tag{9}$$

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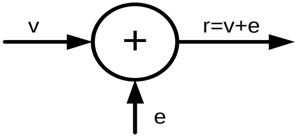

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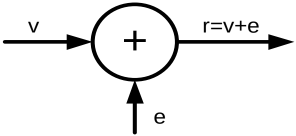

- The decoder first determine whether r contains errors.
- If errors are detected, correct errors (FEC) or Request for a retransmission of v (ARQ).

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- When the error pattern e is identical to a nonzero code word (i.e., r contain errors but $s = r \cdot H^T = 0$), error patterns of this kind are called undetectable error patterns
- Since there are 2^{k-1} nonzero code words, there are 2^{k-1} undetectable error patterns

$$S = (r_0, r_1, \ldots, r_{n-1}).H^T = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \ldots & 1 \\ p_{00} & p_{01} & p_{02} & \cdots & p_{0,n-k-1} \\ p_{10} & p_{11} & p_{12} & \cdots & p_{1,n-k-1} \\ \vdots & & & & \\ p_{k-1,0} & p_{k-1,1} & p_{k-1,2} & \cdots & p_{k-1,n-k-1} \end{bmatrix}$$

- Based on Equation 10, the syndrome digits are as follows:

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$$s_0 = r_0 + r_{n-k}p_{00} + r_{n-k+1}p_{10} + \ldots + r_{n-1}p_{k-1,0}$$
$$s_1 = r_1 + r_{n-k}p_{01} + r_{n-k+1}p_{11} + \ldots + r_{n-1}p_{k-1,1}$$

$$s_{n-k-1} = r_{n-k-1} + r_{n-k}p_{0,n-k-1} \ldots + r_{n-1}p_{k-1,n-k-1} \tag{11}$$

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- The syndrome s is the vector sum of the received parity digits $(r_0, r_1, \ldots, r_{n-k-1})$ and the parity-check digits recomputed from the received information digits $(r_{n-k}, r_{n-k+1} \ldots, r_{n1})$
- A general syndrome circuit is shown in Fig. 5

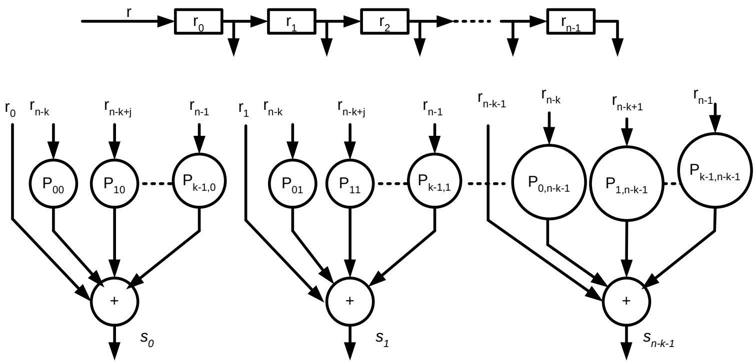

Figure: Syndrome circuit for a liner system (n,k) code

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$$S = (s_0, s_1, s_2) = (r_0, r_1, r_2, r_3, r_4, r_5, r_6) \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

- $s_0 = r_0 + r_3 + r_5 + r_6$
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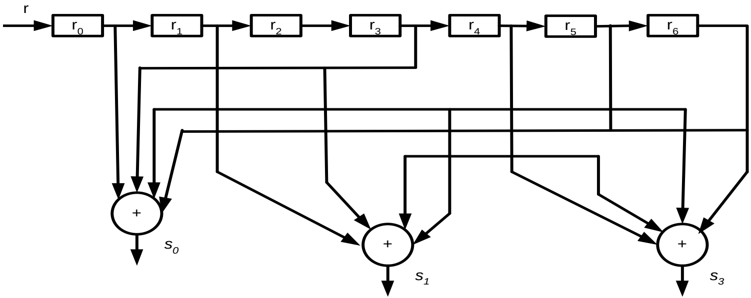
- The syndrome circuit for this code is shown below



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$$s_1 = e_1 + e_{n-k}p_{01} + e_{n-k+1}p_{11} + \ldots + e_{n-1}p_{k-1,1}$$

$$s_{n-k-1} = e_{n-k-1} + e_{n-k}p_{0,n-k-1} \ldots + e_{n-1}p_{k-1,n-k-1} \tag{13}$$

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- There are $2^4 = 16$ error patterns that satisfy the equations above.

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- where v* is the actual transmitted code word

The Minimum Distance of a Block Code

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$$d(v, w) + d(w, x) \geq d(v, x) \qquad (3.14) \qquad (14)$$

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Theorem 3.1

- *The minimum distance of a linear block code is equal to the minimum weight of its nonzero code words.*

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Theorem 3.1

- *The minimum distance of a linear block code is equal to the minimum weight of its nonzero code words.*
- The (7,4) code has minimum weight of 3.

Theorem 3.2

- Let C be an (n, k) linear code with parity-check matrix H. For each code vector of Hamming weight l, there exist l columns of H such that the vector sum of these l columns is equal to the zero vector. Conversely, if there exist l columns of H whose vector sum is the zeros vector, there exists a code vector of Hamming weight l in C.

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- It following from (3.18) that $x.H^T = 0$, x is code vector of weight l in C

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- The random-error-detecting capability of a block code with minimum distance d_{min} is d_{min}-1.

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- If $p = 10^{-2}$ then $P_u(E) = 7x10^{-6}$ this means, if 1 million codewords are transmitted over a BSC with $p = 10^{-2}$ on average seven erroneous codewords pass through the decoder without being detected.

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- The inequality above says that if an error pattern of t or fewer errors occurs, the received vector r is closer (in Hamming distance) to the transmitted code vector v than to any other code vector w in C.
- For a BSC, this means that the conditional probability $P(r|v)$ is greater than the conditional probability $P(r|w)$ for $w \neq v$.

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$$w(e_1) + w(e_2) = w(v + w) = d(v, w) = d_{min}.(3.23) \qquad (22)$$

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$$d(w, r) = w(w + r) = w(w + v + e_1) = w(e_2)(3.25) \qquad (24)$$

 Error-Detecting and Error-Correcting Capabilities of a Block Code

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- Since $2t + 1 \leq d_{min} 2t + 2$, it follows from (3.23) that

$$w(e_2) = d_{min} - w(e_1) \leq (2t + 2) - (t + 1) = t + 1$$

- Combining (3.24) and (3.25) and using the fact that $w(e1) \geq t + 1$ and $w(e2) \leq t + 1$, we have

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- This inequality say that there exists an error pattern of $l(l > t)$ errors which results in a received vector that is closer to an incorrect code vector than to the transmitted code vector.
- Based on the maximum likelihood decoding scheme, an incorrect decoding would be committed.

- A block code with minimum distance d_{min} guarantees correcting all the error patterns of $t = [(d_{min} - 1)/2]$ or fewer errors, where $[(d_{min} - 1)/2]$ denotes the largest integer no greater than $(d_{min} - 1)/2$

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- A block code with random-error-correcting capability t is usually capable of correcting many error patterns of $t + 1$ or more errors.
- For a t-error-correcting (n, k) linear code, it is capable of correcting a total 2^{n-k} error patterns.

Standard Array and Syndrome Decoding

- Let $V_1, V_2, V_3, \ldots, V_{2^k}$ be the code vector of C i.e $C = \{V_1, V_2, , \ldots, V_{2^k}\}$. Each code vector i.e for example $V_1 = (v_0, v_1, \ldots, v_{n-1})$
- Any decoding scheme used at the receiver is a rule to partition the 2^n possible received vectors into 2^k disjoint subsets $D_1, D_2, \ldots, D_{2^k}$ such that the code vector v_i is contained in the subset D_i for $1 \leq i \leq 2^k$.
- Each subset D_i is one-to-one correspondence to a code vector v_i.
- If the received vector r is found in the subset D_i, r is decoded into v_i.
- Correct decoding is made if and only if the received vector r is in the subset D_i that corresponds to the actual code vector transmitted.

- A method to partition the 2^n possible received vectors into 2^k disjoint subsets such that each subset contains one and only one code vector is described here.

  Step 1.
- First, the 2^k code vectors of C are placed in a row with the all-zero code vector $v_1 = (0, 0, \ldots, 0)$ as the first (leftmost) element.

  $D_1, \quad D_2, \ldots, \quad D_i, \quad D_{2}k$

  $v_1 = (00...0) \quad v_2, \ldots, \quad v_i, \quad v_{2}k$

  Step 2.
- From the remaining $2^n - 2^k$ n-tuple, an n-tuple e_2 of minimum weight is chosen and is placed under the zero vector v_1.
- A second row is formed by adding e_2 to each code vector v_i in the first row and placing the sum $e_2 + v_i$ under v_i

Step 3.

- An unused n-tuple e_3 is chosen from the remaining n-tuples and is placed under e_2.
- Then a third row is formed by adding e_3 to each code vector v_i in the first row and placing $e_3 + v_i$ under v_i.
- Continue this process until all the n-tuples are used.
- Then we have an array of rows and columns as shown in Fig 3.6
- This array is called a standard array of the given linear code C

$$
\begin{array}{llllll}
v_1 = 0 & v_2 & \ldots & v_i & \ldots & v_{2^k} \\
e_2 & e_2 + v_2 & \ldots & e_2 + v_i & \ldots & e_2 + v_{2^k} \\
e_3 & e_3 + v_2 & \ldots & e_3 + v_i & \ldots & e_3 + v_{2^k} \\
\vdots & & & & & \\
e_l & e_l + v_2 & \ldots & e_l + v_i & \ldots & e_l + v_{2^k} \\
\vdots & & & & & \\
e_2^{n-k} & e_2^{n-k} + v_2 & \ldots & e_2^{n-k} + v_i & \ldots & e_2^{n-k} + v_{2^k}
\end{array}
$$

Theorem 3.3: *No two n-tuples in the same row of a standard array are identical. Every n-tuple appears in one and only one row.* **Proof:**

- The first part of the theorem follows from the fact that all the code vectors of C are distinct
- Suppose that two n-tuples in the l^{th} rows are identical, say $e_l + v_i = e_l + v_j$ with $i \neq j$
- This means that $v_i = v_j$, which is impossible, therefore no two n-tuples in the same row are identical

Proof

- It follows from the construction rule of the standard array that every n-tuple appears at least once
- Suppose that an n-tuple appears in both lth row and the mth row with $l < m$
- Then this n-tuple must be equal to el + vi for some i and equal to $e_m + v_j$ for some j
- As a result, $e_l + v_i = e_m + v_j$
- From this equality we obtain $em = el + (vi + vj)$
- Since vi and vj are code vectors in C, $vi + v$ is also a code vector in C, say vs
- This implies that the n-tuple em is in the lth row of the array, which contradicts the construction rule of the array that em, the first element of the mth row, should be unused in any previous row
- No n-tuple can appear in more than one row of the array

- From Theorem 3.3 we see that there are $2^n/2^k = 2^{n-k}$ disjoint rows in the standard array, and each row consists of 2^k distinct elements
- The 2_{n-k} rows are called the cosets of the code C
- The first n-tuple e_j of each coset is called a coset leader
- Any element in a coset can be used as its coset leader

- Consider the (6, 3) linear code generated by the following matrix:

$$G = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Message is of

u_0	u_1	u_2
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

The coded message is of $U.G =$
(000000, 011100,101010, 110001, 110110, 101101, 011011, 000111)
The standard array of this code is shown in Table.

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Coset leader							
000000	011100	101010	110001	110110	101101	011011	000111
100000	111100	001010	010001	010110	001101	111011	100111
010000	001100	111010	100001	100110	111101	001011	010111
001000	010100	100010	111001	111110	100101	010011	001111
000100	011000	101110	110101	110010	101001	011111	000011
000010	011110	101000	110011	110100	101111	011001	000101
000001	011101	101011	110000	110111	101100	011010	000110
100100	111000	001110	010101	010010	001001	111111	100011

Standard Array Decoding

- Consider (011100) is the transmitted codeword and the received word is (001100) which lies in 2^{nd} column whose coset leader e=(010000). So e is correctable error pattern. v=r+e= (001100)+(010000)=(011100).
- Consider (011100) is the transmitted codeword and the received word is (010100) which lies in 2^{nd} column whose coset leader e=(001000). So e is correctable error pattern. v=r+e= (010100)+(001000)=(011100).
- Consider (011100) is the transmitted codeword and the received word is (001010) which lies in 2^{nd} column whose coset leader e=(100000). v=r+e= (001010)+(100000)=(101010), in which there are 3 errors ocuur in the received vector that is equal to d_{min}, hence it is undetectable.
- Again consider (011100) is the transmitted codeword and the received word is (101100) and for this error pattern there is no coset leader in the standard array, so e is uncorrectable error pattern.

- A standard array of an (n, k) linear code C consists of 2^k disjoint columns
- Let Dj denote the jth column of the standard array, then

$$D_j = \{v_j, e_2 + v_j, e_3 + v_j, \ldots, e_{2^{n-k}} + v_j\} \qquad (3.27)$$

- v_j is a code vector of C and $e_2, e_3, \ldots e_{2^{n-k}}$ are the coset leaders
- The 2^k disjoint columns $D_1, D_2, \ldots, D_{2^k}$ can be used for decoding the code C.
- Suppose that the code vector v_j is transmitted over a noisy channel, from (3.27) we see that the received vector r is in D_j if the error pattern caused by the channel is a coset leader
- If the error pattern caused by the channel is not a coset leader, an erroneous decoding will result

- The decoding is correct if and only if the error pattern caused by the channel is a coset leader
- The 2^{n-k} coset leaders (including the zero vector 0) are called the correctable error patterns.
  Theorem 3.4 *Every (n, k) linear block code is capable of correcting 2^{n-k} error pattern.*
- To minimize the probability of a decoding error, the error patterns that are most likely to occur for a given channel should be chosen as the coset leaders
- When a standard array is formed, each coset leader should be chosen to be a vector of least weight from the remaining available vectors

Syndrome Decoding

- The syndrome of an n-tuple is an (n-k)-tuple and there are 2^{n-k} distinct (n-k)-tuples.
- From theorem 3.6 that there is a one-to-one correspondence between a coset and an (nk)-tuple syndrome
- Using this one-to-one correspondence relationship, we can form a decoding table, which is much simpler to use than a standard array
- The table consists of 2^{n-k} coset leaders (the correctable error pattern) and their corresponding syndromes
- This table is either stored or wired in the receiver

The decoding of a received vector consists of three steps:
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$$S = r.H^T = H^T.r$$

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Step 2.

- Locate the coset leader e_l whose syndrome is equal to $r.H^T$, then e_l is assumed to be the error pattern caused by the channel.
  Step 3.
- Decode the received vector r into the code vector v. i.e., $v = r + e_l$
- The decoding scheme described above is called the syndrome decoding or table-lookup decoding

Example 3.8

- Consider the (7, 4) linear code given in Table 3.1, the parity-check matrix is given in example 3.3
- The code has $2^3 = 8$ cosets.
- There are eight correctable error patterns (including the all-zero vector)
- Since the minimum distance of the code is 3, it is capable of correcting all the error patterns of weight 1 or 0
- All the 7-tuples of weight 1 or 0 can be used as coset leaders.
- The number of correctable error pattern guaranteed by the minimum distance is equal to the total number of correctable error patterns.

Table: Decoding table for the (7,4) linear code.

Syndrome	**Coset Leader**
(100)	(1000000)
(010)	(0100000)
(001)	(0010000)
(110)	(0001000)
(011)	(0000100)
(111)	(0000010)
(101)	(0000001)

- Suppose that the code vector v = (1 0 0 1 0 1 1) is transmitted and r = (1 0 0 1 1 1 1) is received code vector.

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$$S = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = (0\ 1\ 1)$$

- From Table 3.2 we find that (0 1 1) is the syndrome of the coset leader e = (0 0 0 0 1 0 0), then r is decoded into

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$$\begin{aligned} v^* &= r + e \\ &= (1001111) + (0000100) \\ &= (1001011) \end{aligned}$$

- which is the actual code vector transmitted
- The decoding is correct since the error pattern caused by the channel is a coset leader.

- Suppose that v = (0 0 0 0 0 0 0) is transmitted and r = (1 0 0 0 1 0 0) is received code vector.

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- From the decoding table we find that the coset leader e = (0 0 0 0 0 1 0) corresponds to the syndrome s = (1 1 1).

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e_0 &= f_0(s_0, s_1, \ldots s_{n-k-1}) \\
e_1 &= f_1(s_0, s_1, \ldots s_{n-k-1}) \\
&\vdots \\
e_{n-1} &= f_{n-1}(s_0, s_1, \ldots s_{n-k-1})
\end{aligned}
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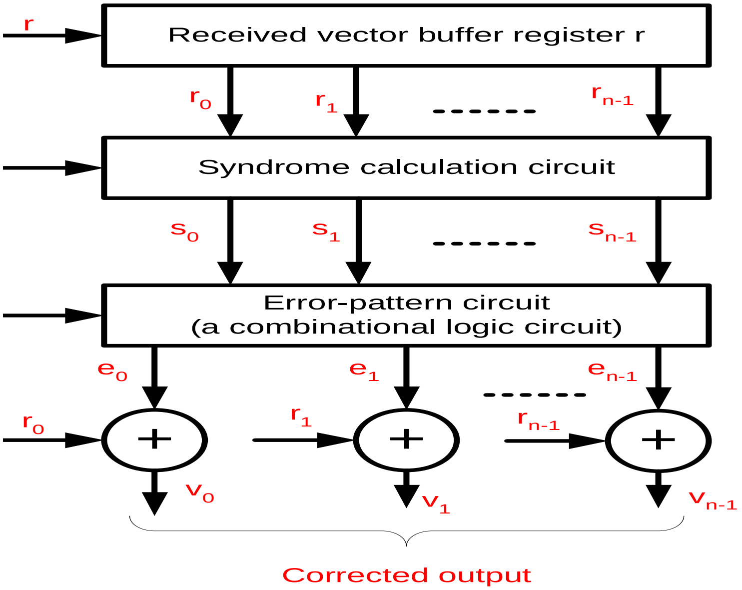
\end{aligned}$$

where $s_0, s_1, \ldots, s_{n-k-1}$ are the syndrome digits where $e_0, e_1, \ldots, e_{n-1}$ are the estimated error digits


r
Received vector buffer register r
r_0
r_1
r_{n-1}
Syndrome calculation circuit
s_0
s_1
s_{n-1}
Error-pattern circuit
(a combinational logic circuit)
e_0
e_1
e_{n-1}
r_0
r_1
r_{n-1}
v_0
v_1
v_{n-1}
Corrected output

Example 3.9

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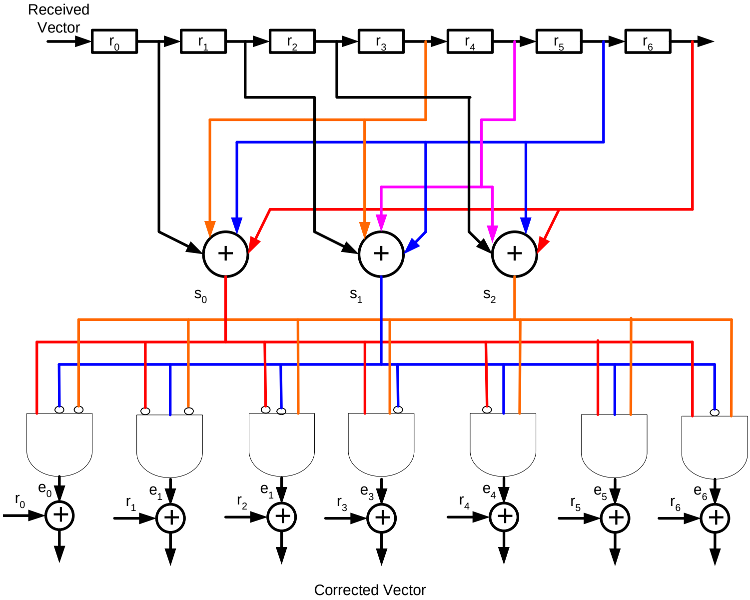
$e_0 = s_0 \wedge s_1^{'} \wedge s_2^{'}$ $\quad e_1 = s_0^{'} \wedge s_1 \wedge s_2^{'}$ $\quad e_2 = s_0^{'} \wedge s_1^{'} \wedge s_2$

$e_3 = s_0 \wedge s_1 \wedge s_2^{'}$ $\quad e_4 = s_0^{'} \wedge s_1 \wedge s_2$ $\quad e_5 = s_0 \wedge s_1 \wedge s_2$

$e_6 = s_0 \wedge s_1^{'} \wedge s_2$


Received Vector
r0
r1
r2
r3
r4
r5
r6
s0
s1
s2
e0
e1
e2
e3
e4
e5
e6
Corrected Vector

Thank You

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