Digital Modulation Techniques:[1, 2, 3, 4]

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Digital Modulation Formats

1. Coherent Binary Modulation Techniques
2. Coherent Quadrature Modulation Techniques
3. Non-Coherent Binary Modulation Techniques

Introduction

Trigonometric identities

$$
\begin{aligned}
cos(x)cos(y) &= 1/2[cos(x-y)+cos(x+y)] \\
sin(x)sin(y) &= 1/2[cos(x-y)-cos(x+y)] \\
sin(x)cos(y) &= 1/2[sin(x+y)+sin(x-y)] \\
cos(x)sin(y) &= 1/2[sin(x+y)-sin(x-y)]
\end{aligned}
$$

$$
\begin{aligned}
sin(x \pm y) &= sin\ x\ cos\ y \pm cos\ x\ sin\ y \\
cos(x \pm y) &= cos\ x\ cos\ y \mp sin\ x\ sin\ y
\end{aligned}
$$

- Modulation is the process by which an information signal is converted to a sinusoidal waveform by varying the amplitude, frequency,or phase or a combination of them of an RF carrier.
- The general form the the carrier wave is

$$s(t) = A(t)\ cos\theta(t) \tag{1}$$

where $A(t)$ is the time varying amplitude and $\theta(t)$ is the time varying angle.

$$s(t) = A(t)\ cos[\omega t + \phi(t)] \tag{2}$$

- where ω is the angular frequency of the carrier and $\phi(t)$ is the phase. The frequency f in in hertz and ω is in radians per second and are related by $\omega = 2\pi f$.

The carrier wave amplitude coefficient

- The general form the the carrier wave is

$$s(t) = A\ cos\omega t \tag{3}$$

- where A is peak value of the waveform. The peak value of the sinusoidal waveform equals $\sqrt{2}$ times the root-mean square(rms) value. Hence

$$s(t) = \sqrt{2}A_{rms}\ cos\omega t = \sqrt{2A_{rms}^2}\ cos\omega t \tag{4}$$

- $\sqrt{2A_{rms}^2}$ represents the average power P normalized to 1Ω. Therefore $s(t) = \sqrt{2P}\ coswt$
- Replacing P watts by E joules/T seconds i.e., $P = E/T$

$$s(t) = \sqrt{\frac{2E}{T}}\ cos\omega t \tag{5}$$

- The energy of a received signal is the key parameter in determining the error performance of the detection process, hence it is often more convenient to use the amplitude notation because it facilitates solving the probability of error P_e as a function of signal energy.

Noise in Communication Systems

- The term noise refers to unwanted electrical signals that are always present in electrical systems.
- The noise arises from a variety of sources, both man made and natural.
- The man made noise includes such sources as **spark-plug ignition noise, switching transients, and other radiating electromagnetic signals**.
- Natural noise includes such elements as the **atmosphere, the sun and other galactic sources,thermal noise or Johnson noise**.
- Noise can be eliminated through filtering, shielding, the choice of modulation and the selection of an optimum receiver.
- **Thermal noise is caused by the thermal motion of electrons in all dissipative components like resistors, wires** and so on.
- Thermal noise cannot be eliminated due to the same electrons are responsible for electrical conduction.
- The **thermal noise can be described by a zero mean Gaussian random process.**
- A Gaussian process $n(t)$ is a random function whose value n at any arbitrary time t is statically characterized by the Gaussian probability density function.

Gaussian, Normal distribution

- In probability theory, the normal (or Gaussian) distribution is a continuous probability distribution, defined by the formula:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- The parameter μ is the mean or expectation of the distribution (and also its median and mode). The parameter σ is its standard deviation; its variance is therefore σ^2.
- A random variable with a Gaussian distribution is said to be normally distributed and is called a normal deviate.
- If $\mu = 0$ and $\sigma = 1$, the distribution is called the standard normal distribution or the unit normal distribution, and a random variable with that distribution is a standard normal deviate.

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x)^2}{2}}$$

- Many things closely follow a Normal Distribution:
  1. Heights of people
  2. Size of things produced by machines
  3. Errors in measurements
  4. Blood pressure
  5. Marks on a test

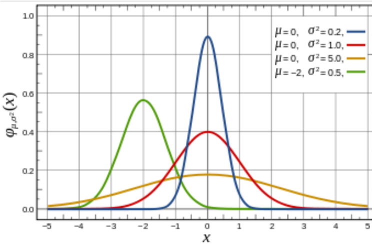


Figure 1: Gaussian Distribution

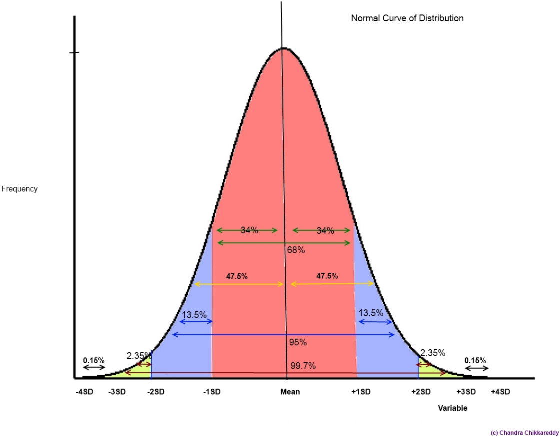


Figure 2: Gaussian Distribution

White Noise

- In signal processing, white noise is a random signal with a flat (constant) power spectral density (PSD).
- White noise draws its name from white light, which is commonly (but incorrectly) assumed to have a flat spectral power density over the visible band.
- The thermal noise is that its PSD is same for all frequencies from dc to 10^{12} Hz.
- The thermal noise is assumed that its PSD $G_n(f)$

$$G_n(f) = N_0/2 \qquad watts/hertz \tag{6}$$

- where the factor 2 is included to indicate that $G_n(f)$ is two sided PSD.
- The autocorrelation function of white noise is given by the inverse Fourier transform of the PSD

$$R_n(\tau) = F\{G_n(f)\} = \frac{N_0}{2}\delta(\tau) \tag{7}$$

- The autocorrelation of white noise is a delta function weighted by the factor $N_0/2$ and occurring at $\tau = 0$ as shown in Figure 3.
- $R_n(\tau) = 0$ for $\tau \neq 0$ that is any two different samples of white noise no matter how close together in time they are uncorrelated.

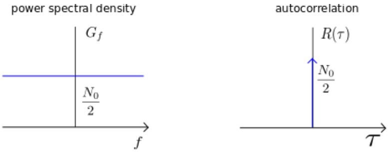


Figure 3: Power spectral density and Autocorrelation of white noise.

Coherent Binary Modulation Techniques

Amplitude Shift Keying (ASK)or On-Off Keying

- The amplitude is varied in according to the input data stream, frequency and phase of the carrier are constant.

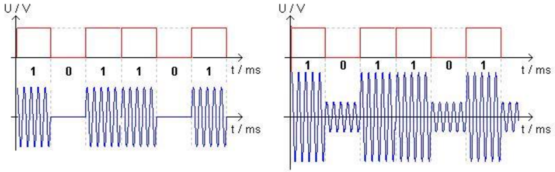


Figure 4: ASK waveform

- In ASK when the symbol is '1' and '0' is given to the product modulator the output is defined as

$$s_1(t) = \sqrt{\frac{2E_b}{T_b}}\cos[2\pi f_c t] \qquad 0 \le t \le T_b \tag{8}$$

$$s_2(t) = 0 \qquad 0 \le t \le T_b \tag{9}$$

where E_b is the transmitted signal energy per bit. $E_b = \frac{A^2 T_b}{2} \quad \therefore \quad A = \sqrt{\frac{2E_b}{T_b}}$

- There is a single basis function $\phi_1(t)$ and is given by:

$$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t) \qquad 0 \leq t \leq T_b$$

$$s_1(t) = \sqrt{E_b}\phi_1(t) \qquad 0 \leq t \leq T_b \tag{10}$$

$$s_2(t) = 0 \qquad 0 \leq t \leq T_b \tag{11}$$

- The source signals are generally referred to as baseband signals.
- The amplitude of a carrier is switched or keyed by the binary signal m(t). This is sometimes called on-off keying (OOK).
- The message signal '0' or '1' is represented by NRZ binary data 0 or $\sqrt{E_b}$ and carrier wave $\sqrt{\frac{2}{T_b}} \cos(2\pi f_c t)$ are given to the modulator.
- The output of the modulator is the ASK modulated signal which may be 0 or $\sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t)$
- The block diagram of ASK (ON/OFF keying) is as shown in Figure 5.

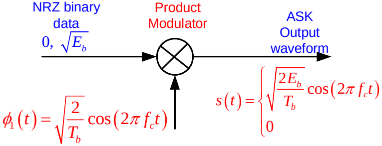


Figure 5: ASK Tranmitter

- ASK system is having a signal space with one dimension and having two message points that have the coordinate points as follows:

$$s_{11} = \int_0^{T_B} s_2(t)\phi_1(t)dt = \int_0^{T_B} \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t)\sqrt{\frac{2}{T_b}} \cos 2\pi f_c t)dt = \int_0^{T_B} \frac{2}{T_b}\sqrt{E_b} \cos^2(2\pi f_c t)dt$$

$$= \frac{2\sqrt{E_b}}{T_b}\int_0^{T_B}\left[\frac{1+cos2(2\pi f_c t)}{2}\right]dt = \frac{\sqrt{E_b}}{T_b}\left[t+\frac{sin2(2\pi f_c t)}{4\pi f_c}\right]_0^{T\_b} = \sqrt{E_b}$$

$$s_{21} = \int_0^{T_B} s_1(t)\phi_1(t)dt = \int_0^{T_B} 0 * \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t)dt = 0$$

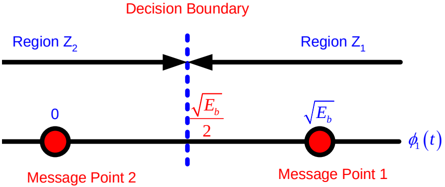


Figure 6: Signal Space diagram

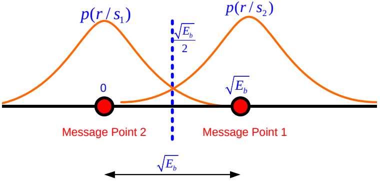


Figure 7: Signal Space diagram

- The message $s_1(t)$ is located at $s_{11} = \sqrt{E_b}$ and $s_2(t)$ is at $s_{21} = 0$.
- The signal space has two regions z_1 and z_2 corresponding to message 1 and 0.
- The distance between message points is $\sqrt{E_b}$.

Error Performance for Digital Modulation

Probability of Bit Error

- An important measure of performance used for comparing digital modulation schemes is the probability of error P_e.
- The probability of the detector making an incorrect decision is termed the probability of symbol error P_e.
- It is often convenient to specify system performance by the probability of bit error P_b even when decisions are made on the basis of symbols for which $M > 2$.

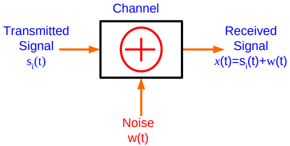


Figure 8: Receiver Model

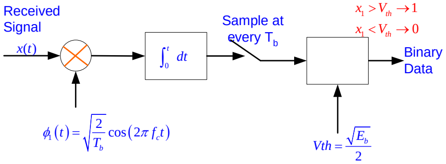


Figure 9: ASK Receiver

- The received signal x(t) in the presence of AWGN w(t)with the assumption that symbol 1 or $s_1(t)$ is transmitted

$$x(t) = s_1(t) + w(t) \qquad 0 \le t \le T_b$$

$$x_1 = \int_0^{Tb} x(t)\phi_1(t)dt = \int_0^{Tb} [s_2(t) + w(t)]\phi_1(t)dt = \int_0^{Tb} s_1(t)\phi_1(t)dt + \int_0^{Tb} w(t)\phi_1(t)dt$$

$$x_1 = s_{11} + w_1 = \sqrt{E_b} + w_1 \qquad \because s_{11} = \sqrt{E_b}$$

- w_1 is the sample value of random variable W_1 having Gaussian distribution with mean zero and variance $N_0/2$ The expected value of the random variable X_1

The expectation (mean) of the random variable X_1

$$E[X_1] = E[s_1] + E[w_1] = E[\sqrt{E_b} + w_1] = \sqrt{E_b} + 0 = \sqrt{E_b} \implies \mu$$

The variance of the random variable X_1

$$Var[X_1] = Var[s_1] + Var[w_1] = 0 + \frac{N_0}{2} = \frac{N_0}{2} \implies \sigma^2$$

Conditional pdf of random variable X_1 given that symbol 1 is transmitted is given by

$$f_{X_1}(x_1|1) = \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{(x_1 - \sqrt{E_b})^2}{N_0}\right] \qquad f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

When symbol 1 is transmitted, an error will occur, if $x_1 < \sqrt{E_b}/2$ in which case a decision is made in favor of symbol 0 $P_e(1) = P(x_1 < \sqrt{E_b}/2|\text{symbol 1 is transmitted})$ $P_e(1)$ can be computed by integrating conditional pdf $f_{X_1}(x_1|1)$

$$P_e(1) = \int_{-\infty}^{\sqrt{E_b}/2} f_{X_1}(x_1|1)dx_1 = \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{\sqrt{E_b}/2} \exp\left[-\frac{(x_1 - \sqrt{E_b})^2}{N_0}\right] dx_1$$

$$Let \quad u = \frac{(x_1 - \sqrt{E_b})}{\sqrt{N_0}}$$

$dx_1 = \sqrt{N_0} du$ Lower limit is $-\infty$ and higher limit is when $x_1 = \sqrt{E_b}/2$ $u = -(1/2)\sqrt{E_b/N_0}$

$$P_e(1) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-(1/2)\sqrt{E_b/N0}} \exp\left[-u^2\right] du$$

$$P_e(1) = \frac{1}{\sqrt{\pi}} \int_{(1/2)\sqrt{E_b/N0}}^{\infty} \exp\left[-u^2\right] du \tag{12}$$

$$erf(u) = \frac{2}{\sqrt{\pi}} \int_0^u \exp(-x^2)dx \quad erfc(u) = 1 - erf(u) = \frac{2}{\sqrt{\pi}} \int_u^{\infty} \exp(-x^2)dx \tag{13}$$

By comparing with complementary error function

$$P_e(1) = \frac{1}{2} erfc\left(\frac{1}{2}\sqrt{\frac{E_b}{N_0}}\right)$$

Similarly when symbol 0 is transmitted, if error occurs, then a decision is made in favor of symbol 10. The probability of error when symbol 0 is transmitted is $P_e(0) = P(x_1 > \sqrt{E_b}/2|$symbol 0 is transmitted. $P_e(0)$ can be computed by integrating conditional pdf $f_{X_1}(x_1|0)$

$$P_e(0) = \frac{1}{2} erfc\left(\frac{1}{2}\sqrt{\frac{E_b}{N_0}}\right)$$

Symbols 0 and 1 are equiprobable, that is $P_e(0) = P_e(1) = 1/2$ then the average probability of symbol error is

$$P_e = \frac{1}{2}[P_e(0) + P_e(1)] = \frac{1}{2} erfc\left(\frac{1}{2}\sqrt{\frac{E_b}{N_0}}\right)$$

$$P_e = \frac{1}{2} erfc\left(\frac{1}{2}\sqrt{\frac{E_b}{N_0}}\right)$$

Binary Phase Shift Keying (BPSK)

- The carrier is shifted in phase according to the input data stream, frequency and amplitude of the carrier are constant.
- Binary PSK (BPSK): two phases (0^o 180^o) represent two binary digits:

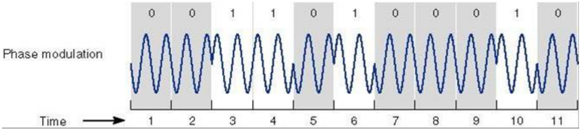


Figure 10: BPSK waveform

- In binary PSK (BPSK) symbols 1 and 0 are represented by $S_1(t)$ $S_2(t)$ and are defined as

$$s_1(t) = \sqrt{\frac{2E_b}{T_b}}\cos[2\pi f_c t + 0^0] = \sqrt{\frac{2E_b}{T_b}}\cos[2\pi f_c t] \qquad 0 \le t \le T_b$$

$$s_2(t) = \sqrt{\frac{2E_b}{T_b}}\cos[2\pi f_c t + 180^0] = -\sqrt{\frac{2E_b}{T_b}}\cos[2\pi f_c t] \qquad 0 \le t \le T_b$$

$$s_i(t) = \sqrt{\frac{2E_b}{T_b}}\cos[2\pi f_c t + \theta_i(t)] \quad i = 1,2 \quad \theta(t) = 0^0 \ or \ 180^0$$

where E_b is the transmitted signal energy per bit. $E_b = \frac{A^2 T_b}{2}$ $\therefore$ $A = \sqrt{\frac{2E_b}{T_b}}$

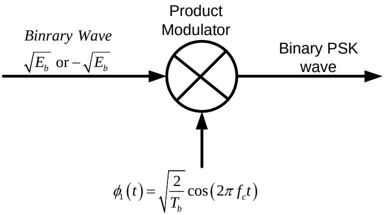


Figure 11: BPSK Transmitter

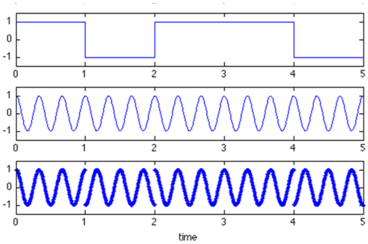


Figure 12: BPSK output waveform

- There is a single basis function $\phi_1(t)$ and is given by:

$$\phi_1(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t) \qquad 0 \le t \le T_b$$

$$\begin{aligned} s_1(t) &= \sqrt{E_b}\phi_1(t) & 0 \le t \le T_b \\ s_2(t) &= -\sqrt{E_b}\phi_1(t) & 0 \le t \le T_b \end{aligned}$$

- BPSK system is having a signal space with one dimension and having two message points that have the coordinate points as follows:

$$s_{11} = \int_0^{T_B} s_1(t)\phi_1(t)dt = \int_0^{T_B} \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t) * \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t)dt = \sqrt{E_b}$$

$$s_{21} = \int_0^{T_B} s_2(t)\phi_1(t)dt = \int_0^{T_B} -\sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t)\sqrt{\frac{2}{T_b}}\cos 2\pi f_c t)dt = -\sqrt{E_b}$$

$$cos^2 x = \frac{1}{2}(1 + cos2x)$$

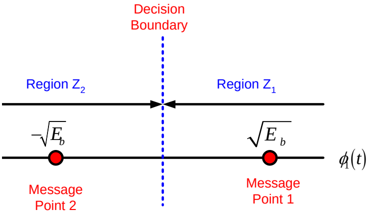


Figure 13: Signal Space diagram

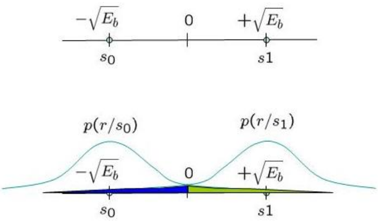


Figure 14: Signal Space diagram

- The message $s_1(t)$ is located at $s_{11} = +\sqrt{E_b}$ $s_2(t)$ is at $s_{21} = -\sqrt{E_b}$.
- The signal space has two regions z_1 and z_2 corresponding to message 1 and 0.
- The distance between message points is $2\sqrt{E_b}$.

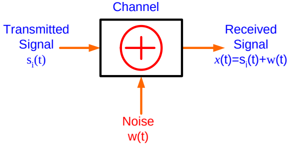


Figure 15: Receiver Model

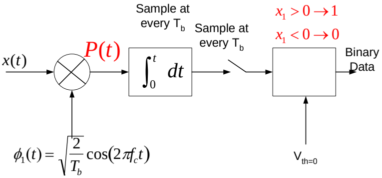


Figure 16: BPSK Receiver

- The received signal x(t) in the presence of AWGN w(t) and by assuming symbol 0 or $s_2(t)$ was transmitted

$$x(t) = s_2(t) + w(t) \qquad 0 \leq t \leq T_b$$

$$\begin{aligned} x_1 &= \int_0^{Tb} x(t)\phi_1(t)dt = \int_0^{Tb} [s_2(t) + w(t)]\phi_1(t)dt \\ &= \int_0^{Tb} s_2(t)\phi_1(t)dt + \int_0^{Tb} w(t)\phi_1(t)dt \\ &= s_{21} + w_1 \end{aligned}$$

$$s_{21} = -\sqrt{E_b}$$

$$x_1 = -\sqrt{E_b} + w_1$$

- w_1 is the sample value of random variable W_1 having Gaussian distribution with mean zero and variance $N_0/2$ The expected value of the random variable X_1

The expectation (mean) of the random variable X_1

$$E[X_1] = E[-\sqrt{E_b} + w_1] = -\sqrt{E_b} + 0 = -\sqrt{E_b} \quad \Longrightarrow \quad \mu$$

The variance of the random variable X_1

$$Var[X_1] = Var[-\sqrt{E_b} + w_1] = 0 + \frac{N_0}{2} = \frac{N_0}{2} \quad \Longrightarrow \quad \sigma^2$$

Conditional pdf of random variable X_1 given that symbol 0 is transmitted is given by

$$f_{X_1}(x_1|0) = \frac{1}{\sqrt{\pi N_0}} \exp\left[\frac{-(x_1 + \sqrt{E_b})^2}{N_0}\right] \qquad f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

When symbol 0 is transmitted, an error will occur, if $x_1 > 0$ in which case a decision is made in favor of symbol 1 $P_e(0) = P(x_1 > 0|\text{symbol 0 is transmitted})$ $P_e(0)$ can be computed by integrating conditional pdf $f_{X_1}(x_1|0)$

$$P_e(0) = \int_0^\infty f_{X_1}(x_1|0)dx_1 = \frac{1}{\sqrt{\pi N_0}} \int_0^\infty \exp\left[\frac{-(x_1 + \sqrt{E_b})^2}{N_0}\right] dx_1$$

$$P_e(0) = \frac{1}{\sqrt{\pi N_0}} \int_0^\infty \exp\left[\frac{-(x_1 + \sqrt{E_b})^2}{N_0}\right] dx_1$$

$$Let \quad u = \frac{(x_1 + \sqrt{E_b})}{\sqrt{N_0}}$$

$dx_1 = \sqrt{N_0}du$ Lower limits when $x = 0$ $u = \sqrt{E_b/N_0}$ and higher limit is ∞

$$P_e(0) = \frac{1}{\sqrt{\pi}} \int_{\sqrt{E_b/N_0}}^\infty \exp\left[-u^2\right] du$$

$$P_e(0) = \frac{1}{\sqrt{\pi}} \int_{\sqrt{E_b/N_0}}^{\infty} \exp\left[-u^2\right] du$$

$$erf(u) = \frac{2}{\sqrt{\pi}} \int_0^u \exp(-x^2) dx$$

$$erfc(u) = 1 - erf(u) = \frac{2}{\sqrt{\pi}} \int_u^{\infty} \exp(-x^2) dx$$

$$P_e(0) = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Similarly when symbol 1 is transmitted, if error occurs, then a decision is made in favor of symbol 0. The probability of error when symbol 1 is transmitted is $P_e(1) = P(x_1 < 0|\text{symbol 1 is transmitted}$. $P_e(0)$ can be computed by integrating conditional pdf $f_{X_1}(x_1|0)$

$$P_e(1) = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$$

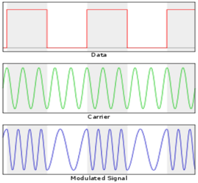
Symbols 0 and 1 are equiprobable, that is $P_e(0) = P_e(1) = 1/2$ then the average probability of symbol error is

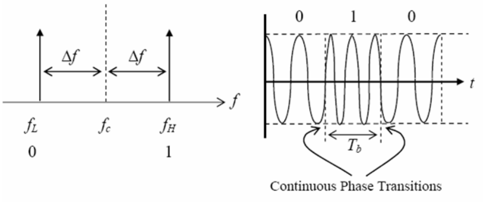
$$P_e = \frac{1}{2}[P_e(0) + P_e(1)] = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$$

$$P_e = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Frequency Shift Keying (FSK)

- The carrier frequency is shifted in according to the input data stream, phase and amplitude of the carrier are constant.





$$s_i(t) = \sqrt{\frac{2E_b}{T_b}} \cos[2\pi f_i t] \qquad\qquad 0 \leq t \leq T_b$$

where i=1,2 and E_b is the transmitted signal energy per bit, and the transmitted frequency $f_i = \frac{n_c + i}{T_b}$ for some fixed integer n_c and i=1,2.

- In binary FSK (BFSK) two different carrier frequencies (f_1 f_2) are used and symbols 1 and 0 are represented by $S_1(t)$ $S_2(t)$ and are defined as

$$s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos[2\pi f_1 t] \quad 0 \leq t \leq T_b \qquad s_2(t) = \sqrt{\frac{2E_b}{T_b}} \cos[2\pi f_2 t] \quad 0 \leq t \leq T_b$$

- There is set of orthonormal basis function $\phi_1(t)$ and $\phi_2(t)$ and is given by:

$$\phi_i(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_i t) \qquad 0 \leq t \leq T_b \qquad i = 1, 2$$

$$s_1(t) = \sqrt{E_b}\phi_1(t) \quad 0 \leq t \leq T_b \qquad s_2(t) = \sqrt{E_b}\phi_2(t) \quad 0 \leq t \leq T_b$$

 Frequency Shift Keying (FSK)

- BFSK system is having a signal space with two dimension and having two message points represented by coordinate points as follows

$$s_{11} = \int_0^{T_B} s_1(t)\phi_1(t)dt = \int_0^{T_B} \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_1 t) * \sqrt{\frac{2}{T_b}}\cos(2\pi f_1 t)dt = \sqrt{E_b}$$

and

$$s_{12} = \int_0^{T_B} s_1(t)\phi_2(t)dt = \int_0^{T_B} \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_1 t) * \sqrt{\frac{2}{T_b}}\cos(2\pi f_2 t)dt = 0$$

$$S_1 = \begin{bmatrix} \sqrt{E_b} \\ 0 \end{bmatrix}$$

Similarly the coefficients of $s_2(t)$ are s_{21} and s_{22}

$$s_{21} = \int_0^{T_B} s_2(t)\phi_1(t)dt = 0$$

and

$$s_{22} = \int_0^{T_B} s_2(t)\phi_2(t)dt = \sqrt{E_b}$$

$$S_2 = \begin{bmatrix} 0 \\ \sqrt{E_b} \end{bmatrix}$$

The dist. betwn., 2 nodes$=\sqrt{\sqrt{E_b}^2 + \sqrt{E_b}^2} = \sqrt{2E_b}$

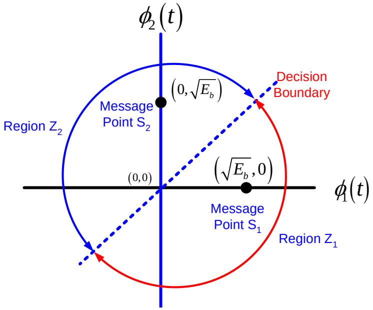


Figure 17: Signal Space diagram for BPSK system

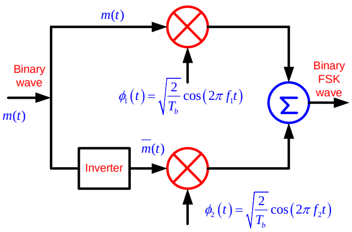


Figure 18: BFSK Transmitter model

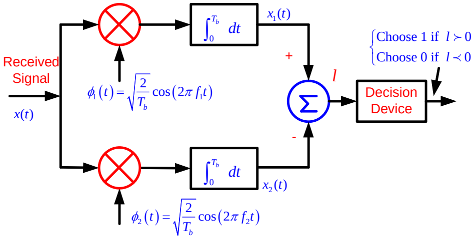


Figure 19: BFSK Receiver model

The observation vector has two elements x_1 and x_2 (message points)which are defined by

$$x_1 = \int_0^{T_B} x(t)\phi_1(t)dt \qquad x_2 = \int_0^{T_B} x(t)\phi_2(t)dt$$

where $x(t)$ is the received signal and when symbol 1 is transmitted

$$x(t) = s_1(t) + w(t)$$

and when symbol 0 is transmitted

$$x(t) = s_2(t) + w(t)$$

where $w(t)$ is the sample function of white Gaussian noise process of zero mean and variance (PSD) is $N_0/2$

Define a new Gaussian random variable L whose sample value l is

$$l = x_1 - x_2$$

The mean value of the random variable L depends on which binary symbol was transmitted. When symbol 1 was transmitted the Gaussian random variables X_1 and X_2 and whose sample values x_1 and x_2 have mean equal to $\sqrt{E_b}$ and zero. The conditional mean value of the random variable L given that symbol 1 was transmitted is given by

$$E[L|1] = E[X_1|1] - E[X_2|1] = \sqrt{E_b}$$

Similarly the conditional mean value of the random variable L given that symbol 0 was transmitted is given by

$$E[L|0] = E[X_1|0] - E[X_2|0] = -\sqrt{E_b} \quad \Longrightarrow \quad \mu$$

Random variables are statistically independent, each with a variance equal to $N_0/2$

$$Var[L] = Var[X_1] + Var[X_2] = N_0/2 + N_0/2 = N_0 \quad \Longrightarrow \quad \sigma^2$$

Suppose when symbol 0 was transmitted, then the conditional value of the conditional probability density function of the random variable L equals

$$f_L(l|0) = \frac{1}{\sqrt{2\pi N_0}} exp\left[-\frac{(l+\sqrt{E_b})^2}{2N_0}\right] \qquad f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Since the condition $x_1 > x_2$ or $l > 0$ then the receiver is making decision in favor of symbol 1 the conditional probability of error, given that symbol 0 was transmitted is given by,

$$P_e(0) = P_e(l > 0| \;\; symbol \;\; 0 \;\; was \;\; sent) = \int_0^\infty f_L(l|0) dl$$

$$P_e(0) = \frac{1}{\sqrt{2\pi N_0}} \int_0^\infty exp\left[-\frac{(l+\sqrt{E_b})^2}{2N_0}\right] dl$$

$$P_e(0) = \frac{1}{\sqrt{2\pi N_0}} \int_0^{\infty} exp\left[-\frac{(l+\sqrt{E_b})^2}{2N_0}\right] dl$$

Let

$$\frac{(l+\sqrt{E_b})}{\sqrt{2N_0}} = z$$

$$dl = \sqrt{2N_0}dz$$

$$P_e(0) = \frac{1}{\sqrt{\pi}} \int_{\sqrt{\frac{E_b}{2N_0}}}^{\infty} exp\left[-z^2\right] dz$$

$$P_e(0) = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{2N_0}}\right)$$

Similarly conditional probability of error $P_e(1)$, when symbol 1 was transmitted has the same value of $P_e(0)$ i.e.,

$$P_e(1) = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{2N_0}}\right)$$

The average probability of symbol error P_e for coherent binary FSK is

$$P_e = \frac{1}{2}P_e(0) + \frac{1}{2}P_e(1) = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{2N_0}}\right)$$

In order to maintain the same average error rate as in coherent BPSK system, the BFSK system should have twice the bit energy to noise density ratio, $\sqrt{E_b}/2N_0$. In coherent BPSK system the distance between the two message points is equal to $2\sqrt{E_b}$ whereas in coherent BPSK system it is $\sqrt{2E_b}$. Larger the distance between the message points smaller is the average probability of error P_e.

Quadriphase Shift Keying (QPSK)

The important goal in design of a digital communication system is to provide low probability of error and efficient utilization of channel bandwidth.

In QPSK, the phase of the carrier takes on one of four equally spaced values, such as $\pi/4,\ 3\pi/4\ 5\pi/4\ 7\pi/4$ as given below

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos\left[2\pi f_c t + (2i-1)\frac{\pi}{4}\right] \qquad 0 \le t \le T$$

where i=1,2,3,4, E is the transmitted signal energy per symbol, T is symbol duration, and the carrier frequency fc equals n_c/T for some fixed integer n_c.

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos\left[(2i-1)\frac{\pi}{4}\right] \cos 2\pi f_c t - \sqrt{\frac{2E}{T}} \sin\left[(2i-1)\frac{\pi}{4}\right] \sin 2\pi f_c t$$

There are two orthonormal basis functions contained in the $s_i(t)$

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t) \qquad 0 \le t \le T$$

$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t) \qquad 0 \le t \le T$$

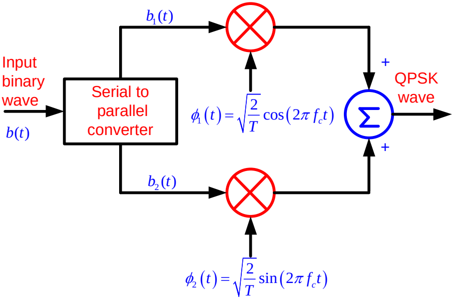


Figure 20: QPSK Transmitter

Constellation Diagram

- A constellation diagram helps us to define the amplitude and phase of a signal when we are using two carriers, one in quadrature of the other.
- The X-axis represents the in-phase carrier and the Y-axis represents quadrature phase carrier.

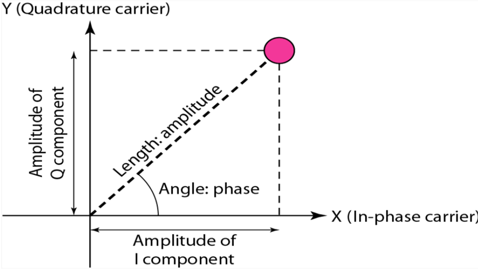


Figure 21: Constellation Diagram

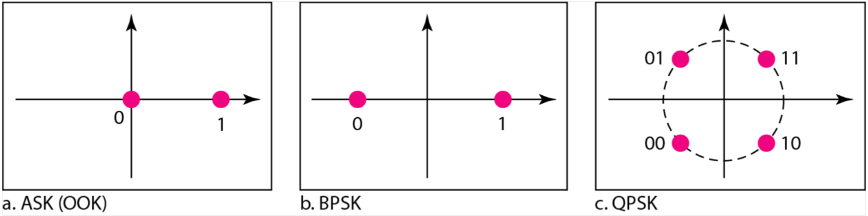


Figure 22: Constellation Diagram

There are four message points

$$S_i = \begin{bmatrix} \sqrt{E}\cos\left((2i-1)\frac{\pi}{4}\right) \\ -\sqrt{E}\sin\left((2i-1)\frac{\pi}{4}\right) \end{bmatrix} \qquad i = 1, 2, 3, 4$$

Input dibit $0 \leq t \leq T$	Phase of QPSK signal	Coordinates of message points	
		s_{i1}	s_{i2}
11	$\pi/4$	$+\sqrt{E/2}$	$+\sqrt{E/2}$
01	$3\pi/4$	$-\sqrt{E/2}$	$+\sqrt{E/2}$
00	$5\pi/4$	$-\sqrt{E/2}$	$-\sqrt{E/2}$
10	$7\pi/4$	$+\sqrt{E/2}$	$-\sqrt{E/2}$

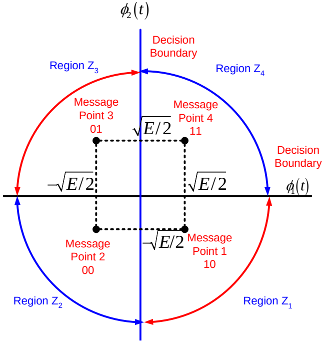
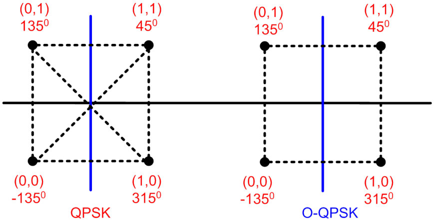


Figure 23: QPSK-signalspace



The received signal is defined by

$$x(t) = s_i(t) + w(t) \qquad 0 \le t \le T$$

$i = 1, 2, 3, 4$

The observation vector x of a coherent QPSK receiver has 2 elements x_1 and x_2 and are defined by

$$x_1 = \int_0^T x(t)\phi_1(t)dt = \sqrt{E}\cos\left[(2i-1)\frac{\pi}{4}\right] + w_1$$

$$x_2 = \int_0^T x(t)\phi_2(t)dt = -\sqrt{E}\sin\left[(2i-1)\frac{\pi}{4}\right] + w_2$$

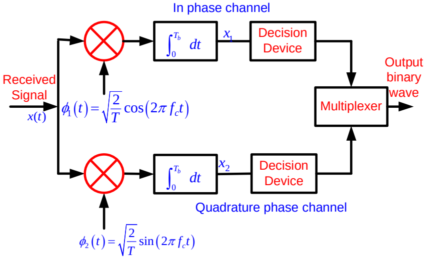


Figure 24: QPSK Receiver model

Suppose signal $s_4(t)$ was transmitted, then the receiver will make correct decision if the observation vector x lies inside the region Z_4. The probability of a correct decision P_c, when signal $s_4(t)$ is transmitted is given by

$$P_c(s_4(t)) = P(x_1 > 0 \;\; and \;\; x_2 > 0)$$

The probability of $P(x_1 > 0 \;\; and \;\; x_2 > 0)$

$$f_{x_1}(x_1|s_4(t)) \;\; transmitted) = \frac{1}{\sqrt{\pi N_0}} exp\left[-\frac{(x_1 - \sqrt{E/2})^2}{N_0}\right]$$

$$f_{x_2}(x_2|s_4(t)) \;\; transmitted) = \frac{1}{\sqrt{\pi N_0}} exp\left[-\frac{(x_2 - \sqrt{E/2})^2}{N_0}\right]$$

$$P_c = \int_0^\infty \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{\left(x_1 - \sqrt{E/2}\right)^2}{N_0}\right] dx_1 . \int_0^\infty \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{\left(x_2 - \sqrt{E/2}\right)^2}{N_0}\right] dx_2$$

$$\frac{\left(x_1 - \sqrt{E/2}\right)}{\sqrt{N_0}} = \frac{\left(x_2 - \sqrt{E/2}\right)^2}{\sqrt{N_0}} = z$$

$$P_c = \left[\frac{1}{\sqrt{\pi}} \int_{-\sqrt{E/2N_0}}^\infty \exp\left(-z^2\right) dz\right]^2$$

From the definition of complementary error function

$$\frac{1}{\sqrt{\pi}} \int_{-\sqrt{E/2N_0}}^\infty \exp\left(-z^2\right) dz = 1 - erfc\left(\sqrt{\frac{E}{2N_0}}\right)$$

$$P_c = \left[1 - \frac{1}{2} erfc\left(\sqrt{\frac{E}{2N_0}}\right)\right]^2$$

$$= 1 - erfc\left(\sqrt{\frac{E}{2N_0}}\right) + \frac{1}{4} erfc^2\left(\sqrt{\frac{E}{2N_0}}\right)$$

The average probability of symbol error for coherent QPSK

$$P_e = 1 - P_c = erfc\left(\sqrt{\frac{E}{2N_0}}\right) - \frac{1}{4}erfc^2\left(\sqrt{\frac{E}{2N_0}}\right)$$

By ignoring the second term probability of symbol error of QPSK

$$P_e = erfc\left(\sqrt{\frac{E}{2N_0}}\right)$$

Signal points s_1,s_2,s_3 and s_4 are symmetrically located in the two dimensional signal space diagram, then

$$P_e(s_4(t)) = P_e(s_3(t)) = P_e(s_2(t)) = P_e(s_1(t)) \tag{14}$$

The probability of occurrence of four messages is an equiprobable, then

$$P_e = \frac{1}{4}[P_e(s_4(t)) + P_e(s_3(t)) + P_e(s_2(t)) + P_e(s_1(t))] \tag{15}$$

In QPSK there are two bits per symbol E = 2Eb

$$P_e = erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$$

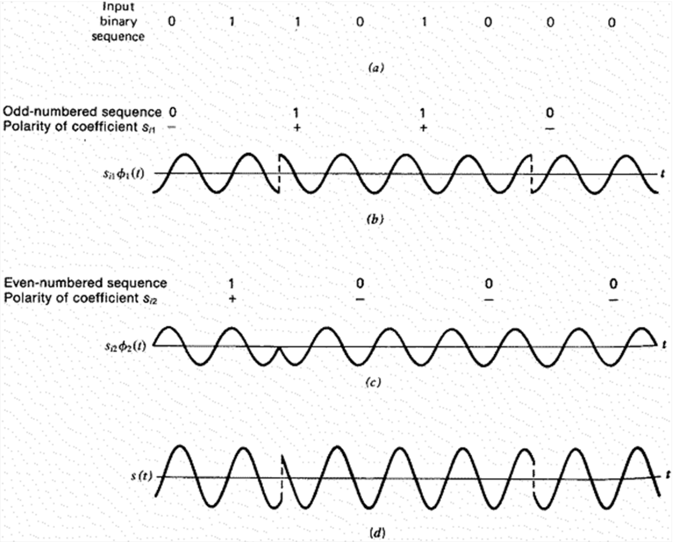


Figure 25: QPSK Waveform

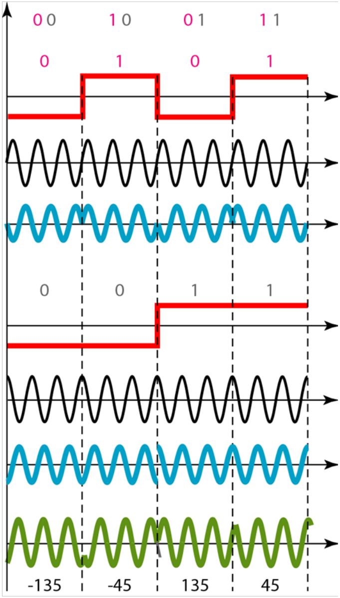


Figure 26: QPSK Waveform

Minimum Shift Keying (MSK)

The performance of the FSK receiver is improved by the proper utilization of phase. Continuous-phase frequency-shift-keying (CPFSK) signal is defined as follows:

$$s(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}} \cos[2\pi f_1 t + \theta(0)] & \textit{for symbol } 1 \\ \sqrt{\frac{2E_b}{T_b}} \cos[2\pi f_2 t + \theta(0)] & \textit{for symbol } 2 \end{cases}$$

E_b is the transmitted signal energy per bit and T_b is the bit duration, phase $\theta(0)$ is the value of the phase at time $t = 0$, depends on the past history of the modulation process. Frequencies f_1 and f_2 represent the symbol 1 and 0 respectively.

The conventional form of an angle modulated wave as follows:

$$s(t) = \sqrt{\frac{2E_b}{T_b}} \cos[2\pi f_c t + \theta(t)]$$

The nominal frequency f_c is chosen as the arithmetic mean of the two frequencies f_1 and f_2

$$f_c = \frac{1}{2}(f_1 + f_2)$$

The phase $\theta(t)$ increases or decreases linearly with time during each bit period of T_b seconds

$$\theta(t) = \theta(0) \pm \frac{\pi h}{T_b} t \qquad 0 \leq t \leq T_b$$

where the plus sign to send symbol 1 and the minus sign to send symbol 0. The parameter h is defined by

$$h = T_b(f_1 - f_2)$$

where h refers to deviation ratio, when $t = T_b$

 Minimum Shift Keying (MSK)

where h refers to deviation ratio, when $t = T_b$

$$\theta(T_b) - \theta(0) = \begin{cases} \pi h \ \textit{for symbol } 1 \\ -\pi h \ \textit{for symbol } 0 \end{cases}$$

To send symbol 1, phase is increased by πh radians whereas to send symbol 0, phase is reduced by πh radians. The phase is an odd or even multiple of πh radians at odd or even multiples of the bit duration T_b respectively. Since all phase shifts are modulo-2 π , the case of $h = 1/2$ is of special case in which phase can take on only two values $\pm\pi/2$ at odd multiples of T_b and only the two values 0 and π at even multiples of T_b. This graph is called a phase trellis.

$$s(t) = \sqrt{\frac{2E_b}{T_b}} \cos[2\pi f_c t + \theta(t)]$$

$$= \sqrt{\frac{2E_b}{T_b}} \cos[\theta(t)] \cos(2\pi f_c t) - \sqrt{\frac{2E_b}{T_b}} \sin[\theta(t)] \sin(2\pi f_c t)$$

Consider the in-phase component $\sqrt{2E_b/T_b} \cos[\theta(t)]$ with deviation $h = 1/2$ then

$$\theta(t) = \theta(0) \pm \frac{\pi}{2T_b} t \qquad 0 \le t \le T_b$$

where the plus sign corresponds to symbol 1 and the minus sign corresponds to symbol 0. A similar results holds for $\theta(t)$ in the interval $-T_b \le t \le 0$. The phase $\theta(0)$ is 0 or π depending on the past history of the modulation process.

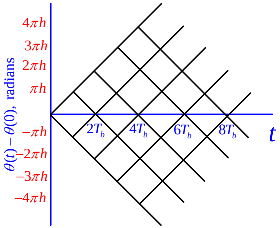


Figure 27: Phase tree

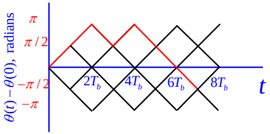


Figure 28: Phase trellis for 1101000

$$\theta(t) = \theta(0) \pm \frac{\pi}{2T_b} t \qquad 0 \le t \le T_b$$

$$s_I(t) = \sqrt{\frac{2E_b}{T_b}} \cos[\theta(t)] = \sqrt{\frac{2E_b}{T_b}} \cos[\theta(0)] \cos\left(\frac{\pi}{2T_b} t\right) = \pm\sqrt{\frac{2E_b}{T_b}} \cos\left(\frac{\pi}{2T_b} t\right)$$

where plus sign corresponds to $\theta(0) = 0$ and the minus corresponds to $\theta(0) = \pi$. Similarly in the interval $0 \le t \le 2T_b$ the quadrature component $s_Q(t)$ consists of half sine pulse and depends on $\theta(T_b)$ and is given by

$$s_Q(t) = \sqrt{\frac{2E_b}{T_b}} \sin[\theta(t)] = \sqrt{\frac{2E_b}{T_b}} \sin[\theta(T_b)] \sin\left(\frac{\pi}{2T_b} t\right) = \pm\sqrt{\frac{2E_b}{T_b}} \sin\left(\frac{\pi}{2T_b} t\right)$$

where plus sign corresponds to $\theta(T_b) = \pi/2$ and the minus corresponds to $\theta(T_b) = -\pi/2$. Consider the equation $h = T_b(f_1 - f_2)$ with $h = 1/2$, then the frequency deviation equals the half the bit rate. This is the minimum frequency spacing in FSK signals to make f_1 and f_2 orthogonal to each other. For this reason CPFSK signal with deviation ratio of one-half is referred to as **minimum shift keying (MSK)**.The phase states of $\theta(0)$ and $\theta(T_b)$ can each assume one of two possible values, and has any four possible combinations and are as follows:

1. The phase $\theta(0) = 0$ and $\theta(T_b) = \pi/2$ to transmit symbol 1.
2. The phase $\theta(0) = \pi$ and $\theta(T_b) = \pi/2$ to transmit symbol 0.
3. The phase $\theta(0) = \pi$ and $\theta(T_b) = -\pi/2$ to transmit symbol 1.
4. The phase $\theta(0) = 0$ and $\theta(T_b) = -\pi/2$ to transmit symbol 0.

$$s(t) = \sqrt{\frac{2E_b}{T_b}} \cos[\theta(t)] \cos(2\pi f_c t) - \sqrt{\frac{2E_b}{T_b}} \sin[\theta(t)] \sin(2\pi f_c t)$$

There are two orthonormal basis functions in $s_i(t)$

$$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos\left(\frac{\pi}{2T_b} t\right) \cos(2\pi f_c t) \qquad \phi_2(t) = \sqrt{\frac{2}{T_b}} \sin\left(\frac{\pi}{2T_b} t\right) \sin(2\pi f_c t)$$

MSK signal is expressed as

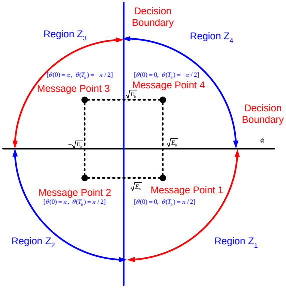
$$s(t) = s_1(t)\phi_1(t) + s_2(t)\phi_2(t) \quad 0 \leq t \leq T_b$$

In phase component of $s(t)$ is

$$s_1(t) = \int_{-T_b}^{T_b} s(t)\phi_1(t) = \sqrt{E_b}\, cos[\theta(0)] \quad -T_b \leq t \leq T_b$$

Quadrature component of $s(t)$ is

$$s_2(t) = \int_0^{2T_b} s(t)\phi_2(t) = -\sqrt{E_b}\, sin[\theta(T_b)] \quad 0 \leq t \leq 2T_b$$



The signal constellation diagram for an MSK signal is of two dimensional with four message points.
The coordinates of the message points are as follows:
$(+\sqrt{E_b}$, $-\sqrt{E_b})$, $(-\sqrt{E_b}$, $-\sqrt{E_b})$, $(-\sqrt{E_b}$, $+\sqrt{E_b})$,$(+\sqrt{E_b}$, $+\sqrt{E_b})$.
The possible values of $\theta(0)$ and $\theta(T_b)$ are as shown in Table

Table 1: Signal Space characterization of MSK

Input bit $0 \leq t \leq T$	Phase states radians		Coordinates of message points	
	$\theta(0)$	$\theta(T_b)$	s_1	s_2
1	0	$+\pi/2$	$+\sqrt{E_b}$	$-\sqrt{E_b}$
0	π	$+\pi/2$	$-\sqrt{E_b}$	$-\sqrt{E_b}$
1	π	$-\pi/2$	$-\sqrt{E_b}$	$+\sqrt{E_b}$
0	0	$-\pi/2$	$+\sqrt{E_b}$	$+\sqrt{E_b}$

Minimum Shift Keying (MSK)

The received signal is defined by: $x(t) = s(t) + w(t)$ where $s(t)$ is the Tx signal, and $w(t)$ is sample value of white Gaussian noise process of zero mean and power spectral density of $N_0/2$. In order to determine whether symbol 1 or 0 was transmitted, determine $\theta(0)$ and $\theta(T_b)$ To determine the phase $\theta(0)$ is as follows

$$x_1(t) = \int_{-T_b}^{T_b} x(t)\phi_1(t)dt = s_1(t)+w(t) \quad -T_b \leq t \leq T_b$$

From the signal space diagram it is observed that if $x_1 > 0$ the receiver decides $\theta(0) = 0$, otherwise when $x_1 < 0$ the receiver decides $\theta(0) = \pi$. Similarly the detection of $\theta(T_b)$ is as follows

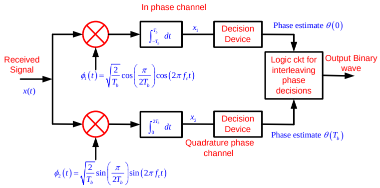


Figure 29: MSK Receiver

$$x_2(t) = \int_{0}^{2T_b} x(t)\phi_2(t)dt = s_2(t)+w(t) \quad 0 \leq t \leq 2T_b$$

If $x_2 > 0$ the received signal decides $\theta(T_b) = -\pi/2$, otherwise when $x_2 < 0$ the receiver decides $\theta(T_b) = \pi/2$. The MSK and QPSK signals have similar signal space-diagram. It follows that the same average probability of symbol error for coherent MSK

$$P_e = 1 - P_c = erfc\left(\sqrt{\frac{E}{2N_0}}\right) - \frac{1}{4}erfc^2\left(\sqrt{\frac{E}{2N_0}}\right)$$

By ignoring the second term probability of symbol error of MSK

$$P_e = erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$$

$$\cos A \cos B = [\cos(A-B) + \cos(A+B)]$$

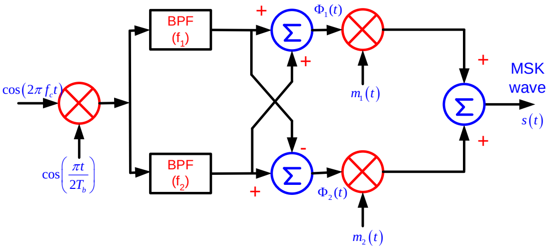
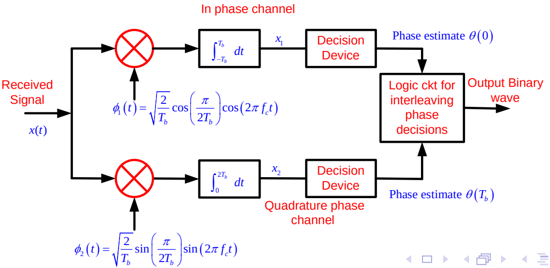


Figure 30: MSK Transmitter



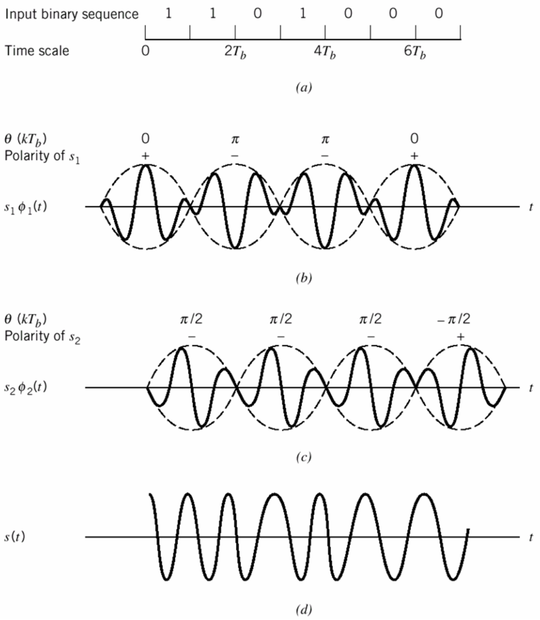


Figure 32: MSK waveform

Non Coherent Modulation Techniques

Non Coherent Binary Frequency Shift Keying (FSK)

In binary FSK the transmitted is defined by:

$$S_i(t) = \begin{cases} \sqrt{\frac{E_b}{2T_b}} cos(2\pi f_i t) & 0 \leq t \leq T_b \\ 0 & elsewhere \end{cases}$$

where the carrier frequency f_i may be f_1 or f_2.
Non coherent FSK receiver consists of a pair of matched filters followed by envelope detectors. The filter in the upper path is matched to $\sqrt{\frac{2}{T_b}} cos(2\pi f_1 t)$ and in the the lower path of the is matched to $\sqrt{\frac{2}{T_b}} cos(2\pi f_2 t)$.
The upper and lower envelope outputs l_1 and l_2 are sampled at $t = T_b$ and their values are compared. If $l_1 > l_2$ the receiver decides in favor of symbol 1 and if $l_1 < l_2$ the receiver decides in favor of symbol 0.

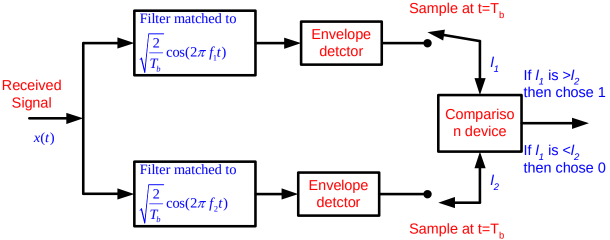


Figure 33: Noncoherent FSK receiver

The noncoherent binary FSK is special case of noncoherent orthogonal modulation with $T = T_b$ and $E = E_b$.
The average probability of error for non coherent FSK is given by

$$P_e = \frac{1}{2} exp\left(-\frac{E_b}{2N_0}\right)$$

Differential Phase Shift Keying (DPSK)

Differential Phase Shift Keying (DPSK) is noncoherent version of the binary PSK system. DPSK eliminates need for a coherent reference signal at the receiver. It consists of differential encoder and phase shift keying. To transmit the symbol 0 the phase is advanced the current signal waveform by 180 and to send symbol 1 the phase of the current signal waveform is unchanged. The differentially encoded sequence d_k is generated by using the logic equation and is given by:

$$d_k = \overline{b_k \oplus d_{k-1}} = d_{k-1} b_k + \overline{d_{k-1}}.\overline{b_k}$$

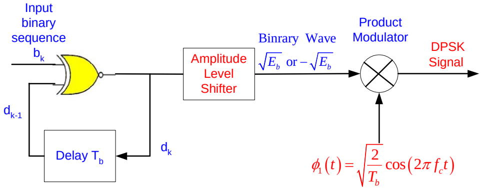


Figure 34: DPSK Transmitter

Table 2: Ex-OR & Ex-NOR Truth Table

A	B	$A \oplus B$	$\overline{A \oplus B}$
0	0	0	1
0	1	1	0
1	0	1	0
1	1	0	1

Table 3: Illustrating the generation of DPSK signal

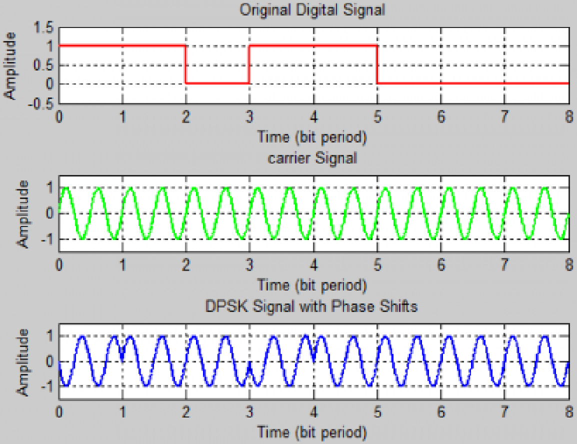
b_k		1	0	0	1	0	0	1	1
d_{k-1}		1	1	0	1	1	0	1	1
d_k	1(Initial assumption)	1	0	1	1	0	1	1	1
Transmitted Phase	0	0	π	0	0	π	0	0	0

Note: π means 180^0 phase shift with respect to unmodulated carrier. Phase shift is 180^0 when the inputs to Ex-Nor gate are dissimilar.

$$d_k = b_k \oplus d_{k-1}$$

b_k		1	1	0	1	1	0	0	0
d_{k-1}		1	0	1	1	0	1	1	1
d_k	1 (Initial assumption)	0	1	1	0	1	1	1	1
Transmitted Phase	0	π	0	0	π	0	0	0	0

Note: π means 180^0 phase shift with respect to unmodulated carrier. Phase shift is 180^0 when the inputs to Ex-Or gate are similar.



The receiver is equipped with a storage capability, so that it measure the relative phase difference between the waveforms received during two successive bit intervals.

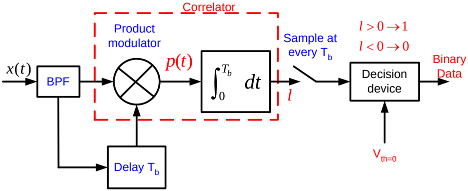


Figure 35: DPSK Receiver

Let $S_1(t)$ denote the transmitted DPSK signal for $0 \leq t \leq 2T_b$ for the case when binary symbol 1 at the transmitter input for the second part of this interval namely $T_b \leq t \leq 2T_b$. The transmission of 1 leaves the carrier phase unchanged, and so $S_1(t)$ is as follows:

$$S_1(t) = \begin{cases} \sqrt{\frac{E_b}{2T_b}} cos(2\pi f_c t) & 0 \leq t \leq T_b \\ \sqrt{\frac{E_b}{2T_b}} cos(2\pi f_c t) & T_b \leq t \leq 2T_b \end{cases} \quad (16)$$

Let $S_2(t)$ denote the transmitted DPSK signal for $0 \leq t \leq 2T_b$ for the case when binary symbol 0 at the transmitter input for the second part of this interval namely $T_b \leq t \leq 2T_b$. The transmission of 0 advances the carrier phase by 180^0, and so $S_2(t)$ is as follows:

$$S_2(t) = \begin{cases} \sqrt{\frac{E_b}{2T_b}} cos(2\pi f_c t) & 0 \leq t \leq T_b \\ \sqrt{\frac{E_b}{2T_b}} cos(2\pi f_c t + \pi) & T_b \leq t \leq 2T_b \end{cases} \quad (17)$$

Let the x(t) is the received signal. The BPF passes only the spectrum of the DBPSK signal and rejects the noise components. In the absence of the noise, the received signal is same as the transmitted signal

$$\sqrt{\frac{E_b}{2T_b}}cos(2\pi f_c t + \theta_i(t)) \quad T_b \leq t \leq 2T_b$$

where i=1,2 and $\theta_i(t) = 0^0 \;\; or \;\; 180^0 \qquad \sqrt{\frac{E_b}{2T_b}}cos(2\pi f_c t)$ or $\quad -\sqrt{\frac{E_b}{2T_b}}cos(2\pi f_c t)$

IF the bits $b_k = d_{k-1}$ (both 0 or 1) then the input to the product modulator are inphase and hence the product modulator output is:

$$p(t) = \frac{E_b}{2T_b}\cos^2[2\pi f_c t] = \frac{E_b}{2T_b}\frac{1}{2}[1 + \cos 2(2\pi f_c t)]$$

at $t = T_b$ the integrator output is:

$$= \int_0^{T_b} \frac{E_b}{4T_b}[1 + \cos 2(2\pi f_c t]dt = E_b/4$$

IF the bits $b_k = 0$ and $d_{k-1} = 1$ or $b_k = 1$ and $d_{k-1} = 0$ then the input to the product modulator are out of phase by π radians hence the product modulator output is:

$$p(t) = -\frac{E_b}{2T_b}\cos^2[2\pi f_c t] = -\frac{E_b}{2T_b}\frac{1}{2}[1 + \cos 2(2\pi f_c t)]$$

at $t = T_b$ the integrator output is:

$$= -\int_0^{T_b} \frac{E_b}{4T_b}[1 + \cos 2(2\pi f_c t+)]dt = -E_b/4$$

Comparison of Binary and quaternary Modulation Techniques

Error Function

In mathematics, the error function (also called the Gauss error function) is a special function (non-elementary) of sigmoid shape which occurs in probability, statistics and partial differential equations. It is defined as

$$erf(u) = \frac{2}{\sqrt{\pi}} \int_0^u \exp(-x^2)dx$$

The complementary error function, denoted *erfc*, is defined as

$$erfc(u) = 1 - erf(u) = \frac{2}{\sqrt{\pi}} \int_u^\infty \exp(-x^2)dx$$

Q Function

In statistics, the Q-function is the tail probability of the standard normal distribution. It is defined as

$$Q(u) = \frac{1}{\sqrt{2\pi}} \int_u^\infty \exp(-x^2/2)dx$$

The Q-function can be expressed in terms of the error function, or the complementary error function, as

$$Q(u) = \frac{1}{2} erfc(u/\sqrt{2}) = \frac{1}{2} - \frac{1}{2} erf(u/\sqrt{2})$$

Table 4: Comparison of Modulation Schemes with Error Probability

Modulation	Detection Method	Error Probability P_e
BPSK	Coherent	$\frac{1}{2} erfc \left(\sqrt{E_b/N_0} \right)$
FSK	Coherent	$\frac{1}{2} erfc \left(\sqrt{E_b/2N_0} \right)$
QPSK	Coherent	$erfc \left(\sqrt{E_b/N_0} \right) - \frac{1}{4} erfc^2 \left(\sqrt{E_b/2N_0} \right)$
MSK	Coherent	$erfc \left(\sqrt{E_b/N_0} \right) - \frac{1}{4} erfc^2 \left(\sqrt{E_b/2N_0} \right)$
DPSK	Non Coherent	$\frac{1}{2} exp \left(-E_b/N_0 \right)$
FSK	Non Coherent	$\frac{1}{2} exp \left(-E_b/2N_0 \right)$

- The error rates for all the systems decease monotonically with increasing values of E_b/N_0.
- Coherent PSK produces a smaller error rate than any of other systems.
- Coherent PSK and DPSK require an E_b/N_0 that is 3 dB less than the corresponding values for conventional coherent FSK and non coherent FSK respectively to realize the same error rate.
- At high values of E_b/N_0 DPSK and noncoherent FSK perform almost as well as coherent PSK and conventional coherent FSK, respectively, for the same bit rate and signal energy per bit
- In QPSK two orthogonal carriers $\sqrt{2/T} cos(2\pi f_c t)$ and $\sqrt{2/T} sin(2\pi f_c t)$ are used, where the carrier frequency f_c is an integral multiple of the symbol rate $1/T$ with the result that two independent bit streams can transmitted and subsequently detected in the receiver. At high values of E_b/N_0 coherently detected binary PSK and QPSK have about the same error rate performance for the same value of E_b/N_0.
- In MSK two orthogonal carriers $\sqrt{2/T_b} cos(2\pi f_c t)$ and $\sqrt{2/T_b} sin(2\pi f_c t)$ are modulated by the two antipodal symbol shaping pulses $cos(\pi t/T_b)$ and $sin(\pi t/T_b)$ respectively over $2T_b$ intervals. Correspondingly, the receiver uses a coherent phase decoding process over two successive bit intervals to recover the original bit stream. MSK has the same error rate performance as QPSK.

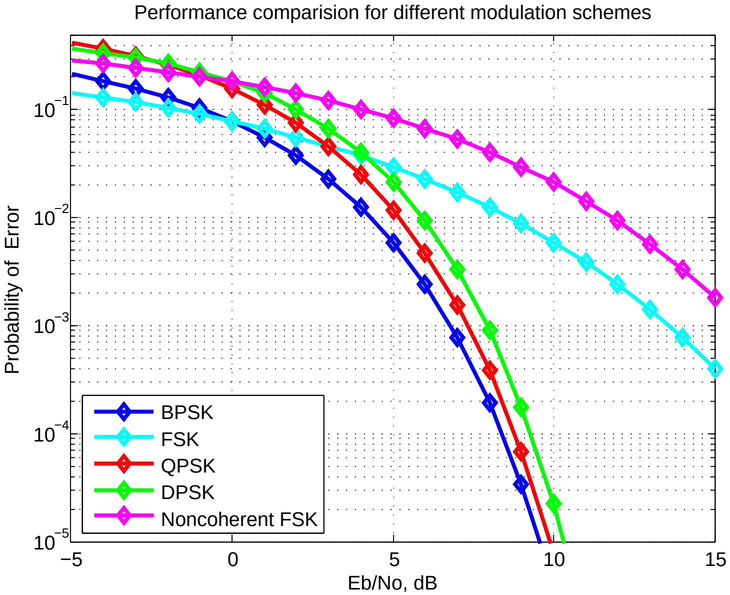


Figure 36: Performance comparision for different modulation schemes

1. With a neat block diagram, explain DPSK transmitter and receiver. Illustrate the generation of differentially encoded sequence for the binary input sequence 00100110011110. July 2016
2. Draw the block diagram for QPSK transmitter and receiver. From the basic principles prove that BER for QPSK is $\frac{1}{2} erfc \left[\sqrt{\frac{E_b}{N_0}} \right]$ July 2015
3. Explain in detail along with block diagram a coherent FSK transmitter and receiver. July 2015
4. Explain generation and demodulation of DPSK wave with block diagram. June 2014
5. Explain briefly phase tree and phase Trellis in MSK June 2014
6. With a neat block diagram, explain DPSK transmitter and receiver.December 2014
7. For the binary sequence 0110100 explain the signal space diagram for coherent QPSK system. December 2014
8. Derive an expression for the average probability of symbol error for coherent binary PSK system. December 2014
9. Derive an expression for probability of error for coherent binary PSK signal. December 2013
10. Obtain the differential encoded sequence and the transmitted phase for the for the binary input data 10010011. December 2013
11. Obtain the expression for the probability of symbol error of coherent binary FSK system. June 2013
12. Compare the probability of error depends on the distance between the message points in signal space diagram. June 2013
13. With a neat block diagram, explain differential phase shift keying. Illustrate the generation of differentially encoded sequence for the binary data 1100100010. June 2013
14. Explain the generation and detection of binary phase shift keying. Dec 2012
15. Find the average probability of symbol error for a coherent QPSK system. Dec 2012

 Comparison of Binary and Quaternary Modulation Techniques

Problems

8.8 A binary FSK system transmits binary data at a rate of 2 MBPS. Assuming channel AWGN with zero mean and power spectral density of $N_0/2 = 1 \times 10^{-20}$ W/Hz. The amplitude of the received signal in the absence of noise is 1 microvolt. Determine the average probability of error for coherent detection of FSK.

Solution:

$$A = \sqrt{\frac{2E_b}{T_b}} \qquad \Rightarrow E_b = \frac{1}{2}A^2 \times T_b$$

Signal energy per bit

$$E_b = \frac{1}{2}A^2 \times T_b = \frac{1}{2}(1 \times 10^{-6})^2 \times 0.5 \times 10^{-6} = 0.25 \times 10^{-18} \; Joulels$$

Given $N_0/2 = 1 \times 10^{-20}$ W/Hz $\qquad \Rightarrow N_0 = 2 \times 10^{-20}$ W/Hz

The average probability of error for coherent PSK is

$$P_e = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{2N_0}}\right) = 0.5 \times erfc\left(\sqrt{\frac{0.25 \times 10^{-18}}{4 \times 10^{-20}}}\right) = 0.5 \times erfc(\sqrt{6.25}) = 0.5 \times erfc(2.5)$$

From error function table

$$erfc(2.5) = 0.000407$$

$$P_e = 0.5 \times 0.000407 = 0.0002035$$

8.4 An FSK system transmits binary data at a rate of 10^6 bits per second. Assuming channel AWGN with zero mean and power spectral density of $N_0/2 = 2 \times 10^{-20}$ W/Hz. Determine the probability of error. Assume coherent detection and amplitude of received sinusoidal signal for both symbol 1 and 0 to be 1.2 microvolt.

Solution:

$$A = \sqrt{\frac{2E_b}{T_b}} \qquad \Rightarrow E_b = \frac{1}{2}A^2 \times T_b$$

Signal energy per bit

$$E_b = \frac{1}{2}A^2 \times T_b = \frac{1}{2}(1.2 \times 10^{-6})^2 \times 10^{-6} = 0.72 \times 10^{-18} \ Joulels$$

Given $N_0/2 = 2 \times 10^{-20}$ W/Hz $\qquad \Rightarrow N_0 = 4 \times 10^{-20}$ W/Hz
The average probability of error for coherent PSK is

$$P_e = \frac{1}{2} erfc \left(\sqrt{\frac{E_b}{2N_0}} \right) = 0.5 \times erfc \left(\sqrt{\frac{0.72 \times 10^{-18}}{8 \times 10^{-20}}} \right) = 0.5 \times erfc \left(\sqrt{0.09 \times 100} \right) = 0.5 \times erfc(3)$$

From complementary error function table

$$erfc(3) = 0.00002$$

$$P_e = 0.5 \times 0.00002 = 0.00001 = 1 \times 10^{-6} =$$

8.4 A binary data are transmitted at a rate of 10^6 bits per second over the microwave link. Assuming channel AWGN with zero mean and power spectral density of 1×10^{-10} W/Hz. Determine the average carrier power required to maintain an average probability of error probability of error $P_e \leq 10^{-4}$ for coherent binary FSK. Determine the minimum channel bandwidth required.

Solution:

The average probability of error for binary FSK is

$$P_e = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{2N_0}}\right) = 10^{-4} \qquad \therefore \quad erfc\left(\sqrt{\frac{E_b}{2N_0}}\right) = 2 \times 10^{-4} = 0.0002$$

From complementary error function table erfc(2.63) = 0.0002

$$\sqrt{\frac{E_b}{2N_0}} = 2 \times 10^{-4} \simeq 2.63 \qquad \therefore \quad \frac{E_b}{2N_0} = 6.9169$$

Given $N_0/2 = 1 \times 10^{-10}$ W/Hz $\qquad \Rightarrow N_0 = 2 \times 10^{-10}$ W/Hz

$$E_b = 6.9169 \times 2 \times 2 \times 10^{-10} = 27.6676$$

The average carrier power required is

$$\frac{E_b}{T_b} = \frac{27.667610^{-10}}{10^{-6}} = 27.667610^{-4}\, Watts$$

The minimum channel bandwidth required approximately is

$$\frac{1}{T_b} = \frac{1}{10^{-6}} = 1MHz$$

8.10 A binary data is transmitted over an AWGN channel using binary PSK at a rate of 1MBPS. It is desired to have average probability of error $P_e \leq 10^{-4}$. Noise power spectral density is $N_0/2 = 1 \times 10^{-12}$ W/Hz. Determine the average carrier power required at the receiver input, if the detector is of coherent type.
July-2016, **July-2016**

Solution:

Let P be the power required at the receiver then $E_b = PT_b$ where T_b is the bit duration.
Since $N_0/2 = 1 \times 10^{-12}$ W/Hz $\quad \Rightarrow N_0 = 2 \times 10^{-12}$ W/Hz

$$T_b = \frac{1}{bit\ rate} = \frac{1}{10^6} = 10^{-6}$$

$$\left(\frac{E_b}{N_0}\right) = \frac{P \times T_b}{N_0} = \frac{P \times 10^{-6}}{2 \times 10^{-12}} = 0.5 \times P \times 10^6$$

The average probability of error for coherent PSK is

$$P_e = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$$

$$P_e = 0.5 \times erfc\sqrt{(0.5 \times P \times 10^6)} \leq \times 10^{-4}$$

$$erfc\sqrt{(0.5 \times P \times 10^6)} \leq 2 \times 10^{-4}$$

From complementary error function table $\quad erfc(2.63) = 0.0002$

$$\sqrt{0.5 \times P \times 10^6} \geq 2.63 \qquad \Rightarrow P \geq 1.315 \times 10^{-6} W$$

Proakis 5.18 Suppose that binary PSK is used for transmitting information over an AWGN with a power spectral density of $N_0/2 = 10^{-10}$ W/Hz. The transmitted signal energy is $E_b = 1/2A^2T$ where T is the bit interval and A is the signal amplitude. Determine the signal amplitude required to achieve an error probability of 10^{-6} when the data rate is a) 10 kbits/s, b) 100 kbits/s, and c)1 Mbits/s

Solution:

$A = \sqrt{\frac{2E_b}{T_b}} \Rightarrow E_b = \frac{1}{2}A^2T_b$ and $P_b = \frac{1}{2}erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$ For binary phase modulation, the error probability is

$$P_b = P_b = \frac{1}{2}erfc\left(\sqrt{\frac{E_b}{N_0}}\right) = \frac{1}{2}erfc\left(\sqrt{\frac{0.5 * A^2T_b}{N_0}}\right)$$

$$P_b = \frac{1}{2}erfc\left(\sqrt{\frac{0.5 * A^2T_b}{N_0}}\right) = 10^{-6} \Rightarrow erfc\left(\sqrt{\frac{0.5 * A^2T_b}{N_0}}\right) = 2 * 10^{-6}$$

We know that $erfc(x) = 2 * 10^{-6}$ and from erfc table the value of x=3.36

$\sqrt{0.5A^2T_b/N_0} = 3.36 \Rightarrow 0.5A^2T_b/N_0 = 11.2896$

$A^2T_b = 22.579N_0 \Rightarrow A^2 = 22.5792 * 10^{-10} * (1/T_b) = 4.515 * 10^{-9} * (1/T_b)$

$A^2 = 4.515 * 10^{-9} * (1/T_b) \Rightarrow A = \sqrt{4.515 * 10^{-9} * (1/T_b)}$

If the data rate is 10 Kbps, then

$$A = \sqrt{4.515 \times 10^{-9} \times 10 \times 10^3} = 6.7193 \times 10^{-3}$$

If the data rate is 100 Kbps, then

$$A = \sqrt{4.515 \times 10^{-9} \times 100 \times 10^3} = 21.2 \times 10^{-3}$$

If the data rate is 1 Mbps, then

$$A = \sqrt{4.515 \times 10^{-9} \times 1 \times 10^6} = 67.19 \times 10^{-3}$$

Problems:Sklar

4.2 A continuously operating coherent BPSK system makes errors at the average rate of 100 errors per day. The data rate is 1000 bits/s. The single-sided noise power spectral density is $N_0 = 10^{-10} W/Hz$.

- If the system is ergodic, what is the average bit error probability?
- If the value of received average signal power is adjusted to be $10^{-6} W$ will, this received power be adequate to maintain the error probability found in part (a)?

Solution:

The total bit detected in one day= $1000 \times 86400 == 8.64 \times 10^7$

$$P_b = \frac{100}{8.64 \times 10^7} = 1.16 \times 10^{-6}$$

$$P_b = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{\frac{2ST}{10^{-10}}}\right)$$

where $S = 10^{-6}$W and $T = \frac{1}{1000}$

$$P_b = Q\left(\sqrt{\frac{2 \times 10^{-6}}{1000 \times 10^{-10}}}\right) = Q(\sqrt{20}) = Q(4.47)$$

Since in Q(x) $x > 3$

$$P_b = \frac{1}{4.47\sqrt{2\Pi}} exp\left(\frac{-(4.47)^2}{2}\right) = 4.05 \times 10^{-6}$$

No

Find the expected number of bit errors made in one day by the following continuously operating coherent BPSK receiver. The data rate is 5000 bps. The input digital waveforms are $s_1(t) = A cos\omega_0 t$ and $s_2(t) = -A cos\omega_0 t$ where A= 1 mV and the the single-sided noise power spectral density is $N_0 = 10^{-11} W/Hz$. Assume that signal power and energy per bit are normalized to a 1 Ω resistive load.

Solution:

$$A = \sqrt{\frac{2E_b}{T_b}} \qquad E_b = \frac{A^2 T_b}{2} \qquad T_b = \frac{1}{R_b}$$

$$P_e = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right) = \frac{1}{2} erfc\left(\sqrt{\frac{A^2 T_b}{2N_0}}\right)$$

$$P_e = \frac{1}{2} erfc\left(\sqrt{\frac{(1 \times 10^{-3})^2}{2 \times 10^{-11} \times 5000}}\right) = \frac{1}{2} erfc(3.1622)$$

$$P_e = \frac{1}{2} \times 0.00000786 = 3.93 \times 10^{-6}$$

Average no of errors in one day $= 5000 bits \times 24 \times 60 \times 60 \times 3.93 \times 10^{-6} \simeq 1698$ bits in error

Complementary Error Function Table													
x	erfc(x)	x	erfc(x)	x	erfc(x)	x	erfc(x)	x	erfc(x)	x	erfc(x)	x	erfc(x)
0	1.000000	0.5	0.479500	1	0.157299	1.5	0.033895	2	0.004678	2.5	0.000407	3	0.00002209
0.01	0.988717	0.51	0.470756	1.01	0.153190	1.51	0.032723	2.01	0.004475	2.51	0.000386	3.01	0.00002074
0.02	0.977435	0.52	0.462101	1.02	0.149162	1.52	0.031587	2.02	0.004281	2.52	0.000365	3.02	0.00001947
0.03	0.966159	0.53	0.453536	1.03	0.145216	1.53	0.030484	2.03	0.004094	2.53	0.000346	3.03	0.00001827
0.04	0.954889	0.54	0.445061	1.04	0.141350	1.54	0.029414	2.04	0.003914	2.54	0.000328	3.04	0.00001714
0.05	0.943628	0.55	0.436677	1.05	0.137564	1.55	0.028377	2.05	0.003742	2.55	0.000311	3.05	0.00001608
0.06	0.932378	0.56	0.428384	1.06	0.133856	1.56	0.027372	2.06	0.003577	2.56	0.000294	3.06	0.00001508
0.07	0.921142	0.57	0.420184	1.07	0.130227	1.57	0.026397	2.07	0.003418	2.57	0.000278	3.07	0.00001414
0.08	0.909922	0.58	0.412077	1.08	0.126674	1.58	0.025453	2.08	0.003266	2.58	0.000264	3.08	0.00001326
0.09	0.898719	0.59	0.404064	1.09	0.123197	1.59	0.024538	2.09	0.003120	2.59	0.000249	3.09	0.00001243
0.1	0.887537	0.6	0.396144	1.1	0.119795	1.6	0.023652	2.1	0.002979	2.6	0.000236	3.1	0.00001165
0.11	0.876377	0.61	0.388319	1.11	0.116467	1.61	0.022793	2.11	0.002845	2.61	0.000223	3.11	0.00001092
0.12	0.865242	0.62	0.380589	1.12	0.113212	1.62	0.021962	2.12	0.002716	2.62	0.000211	3.12	0.00001023
0.13	0.854133	0.63	0.372954	1.13	0.110029	1.63	0.021157	2.13	0.002593	2.63	0.000200	3.13	0.00000958
0.14	0.843053	0.64	0.365414	1.14	0.106918	1.64	0.020378	2.14	0.002475	2.64	0.000189	3.14	0.00000897
0.15	0.832004	0.65	0.357971	1.15	0.103876	1.65	0.019624	2.15	0.002361	2.65	0.000178	3.15	0.00000840
0.16	0.820988	0.66	0.350623	1.16	0.100904	1.66	0.018895	2.16	0.002253	2.66	0.000169	3.16	0.00000786
0.17	0.810008	0.67	0.343372	1.17	0.098000	1.67	0.018190	2.17	0.002149	2.67	0.000159	3.17	0.00000736
0.18	0.799064	0.68	0.336218	1.18	0.095163	1.68	0.017507	2.18	0.002049	2.68	0.000151	3.18	0.00000689
0.19	0.788160	0.69	0.329160	1.19	0.092392	1.69	0.016847	2.19	0.001954	2.69	0.000142	3.19	0.00000644
0.2	0.777297	0.7	0.322199	1.2	0.089686	1.7	0.016210	2.2	0.001863	2.7	0.000134	3.2	0.00000603
0.21	0.766478	0.71	0.315335	1.21	0.087045	1.71	0.015593	2.21	0.001776	2.71	0.000127	3.21	0.00000564
0.22	0.755704	0.72	0.308567	1.22	0.084466	1.72	0.014997	2.22	0.001692	2.72	0.000120	3.22	0.00000527
0.23	0.744977	0.73	0.301896	1.23	0.081950	1.73	0.014422	2.23	0.001612	2.73	0.000113	3.23	0.00000493
0.24	0.734300	0.74	0.295322	1.24	0.079495	1.74	0.013865	2.24	0.001536	2.74	0.000107	3.24	0.00000460
0.25	0.723674	0.75	0.288845	1.25	0.077100	1.75	0.013328	2.25	0.001463	2.75	0.000101	3.25	0.00000430
0.26	0.713100	0.76	0.282463	1.26	0.074764	1.76	0.012810	2.26	0.001393	2.76	0.000095	3.26	0.00000402
0.27	0.702582	0.77	0.276179	1.27	0.072486	1.77	0.012309	2.27	0.001326	2.77	0.000090	3.27	0.00000376
0.28	0.692120	0.78	0.269990	1.28	0.070266	1.78	0.011826	2.28	0.001262	2.78	0.000084	3.28	0.00000351
0.29	0.681717	0.79	0.263897	1.29	0.068101	1.79	0.011359	2.29	0.001201	2.79	0.000080	3.29	0.00000328
0.3	0.671373	0.8	0.257899	1.3	0.065992	1.8	0.010909	2.3	0.001143	2.8	0.000075	3.3	0.00000306
0.31	0.661092	0.81	0.251997	1.31	0.063937	1.81	0.010475	2.31	0.001088	2.81	0.000071	3.31	0.00000285
0.32	0.650874	0.82	0.246189	1.32	0.061935	1.82	0.010057	2.32	0.001034	2.82	0.000067	3.32	0.00000266
0.33	0.640721	0.83	0.240476	1.33	0.059985	1.83	0.009653	2.33	0.000984	2.83	0.000063	3.33	0.00000249
0.34	0.630635	0.84	0.234857	1.34	0.058086	1.84	0.009264	2.34	0.000935	2.84	0.000059	3.34	0.00000232
0.35	0.620618	0.85	0.229332	1.35	0.056238	1.85	0.008889	2.35	0.000889	2.85	0.000056	3.35	0.00000216
0.36	0.610670	0.86	0.223900	1.36	0.054439	1.86	0.008528	2.36	0.000845	2.86	0.000052	3.36	0.00000202
0.37	0.600794	0.87	0.218560	1.37	0.052688	1.87	0.008179	2.37	0.000803	2.87	0.000049	3.37	0.00000188
0.38	0.590991	0.88	0.213313	1.38	0.050984	1.88	0.007844	2.38	0.000763	2.88	0.000046	3.38	0.00000175
0.39	0.581261	0.89	0.208157	1.39	0.049327	1.89	0.007521	2.39	0.000725	2.89	0.000044	3.39	0.00000163
0.4	0.571608	0.9	0.203092	1.4	0.047715	1.9	0.007210	2.4	0.000689	2.9	0.000041	3.4	0.00000152
0.41	0.562031	0.91	0.198117	1.41	0.046148	1.91	0.006910	2.41	0.000654	2.91	0.000039	3.41	0.00000142
0.42	0.552532	0.92	0.193232	1.42	0.044624	1.92	0.006622	2.42	0.000621	2.92	0.000036	3.42	0.00000132
0.43	0.543113	0.93	0.188437	1.43	0.043143	1.93	0.006344	2.43	0.000589	2.93	0.000034	3.43	0.00000123
0.44	0.533775	0.94	0.183729	1.44	0.041703	1.94	0.006077	2.44	0.000559	2.94	0.000032	3.44	0.00000115
0.45	0.524518	0.95	0.179109	1.45	0.040305	1.95	0.005821	2.45	0.000531	2.95	0.000030	3.45	0.00000107
0.46	0.515345	0.96	0.174576	1.46	0.038946	1.96	0.005574	2.46	0.000503	2.96	0.000028	3.46	0.00000099
0.47	0.506255	0.97	0.170130	1.47	0.037627	1.97	0.005336	2.47	0.000477	2.97	0.000027	3.47	0.00000092
0.48	0.497250	0.98	0.165769	1.48	0.036346	1.98	0.005108	2.48	0.000453	2.98	0.000025	3.48	0.00000086
0.49	0.488332	0.99	0.161492	1.49	0.035102	1.99	0.004889	2.49	0.000429	2.99	0.000024	3.49	0.00000080

References

[1] S. Haykin, *Digital communications*, 2nd ed. Wiley, 1988.

[2] B. Sklar and P. K. Ray, *Digital Communications: Fundamentals and Applications*, 2nd ed. Pearson Education, 2001.

[3] J. G. Proakis, *Digital communications*, 4th ed. Prentice Hall, 2001.

[4] J. G. Proakis and M. Salehi, *Communication Systems Engineering*, 2nd ed. Prentice Hall, 2002.

[5] K. N. H. Bhat and D. G. Rao, *Digital Communications: A Simppified Approach*, 2nd ed. Sanguine Technical Publishers, 2005.