Chapter 1

Single Random Variables

1.1 Introduction

Probability: ***Probability is a branch of mathematics which deals with calculating the likelihood of a given event's occurrence, which is expressed as a number between 1 and 0.*** The outcome of a random event cannot be predetermined before it occurs, but it is a any one of all possible outcomes.

$$0 \leq P(A) \leq 1$$

Sample Space S: The set of all possible outcomes of a random phenomenon.

$$P(S) = 1$$

The impossible event is the event in which there is no outcome which is denoted as ϕ

$$P(\phi) = 0$$

Examples: Tossing a Coin: When a coin is tossed, there are two possible outcomes: head (H) or tail (T). The sample space is {H,T}

The probability of any one of them is $\frac{1}{2}$

Throwing Dice: When a single die is thrown, there are six possible outcomes: 1, 2, 3, 4, 5, 6. The sample space is {1, 2, 3, 4, 5, 6}. The probability of any one of them is $\frac{1}{6}$

Properties of Probability:

1. **Axiom I:** The probability that at least one of the event in the entire sample space will occur is 1.

$$P(S) = 1$$

2. **Axiom II:** The probability of an event is a non-negative real number:

$$P(A) \geq 0$$

3. **Axiom III:** If A and B are both contained in the sample space, and if A and B are disjoint events ($A \cap B = \phi$) then

$$P(A \cup B) = P(A) + P(B)$$

Event: Any set of outcomes of interest.

Probability of an event: The relative frequency of the set of outcomes over an infinite number of trials. $P_r(A)$ is the probability of event A

$$P_r(A) = \frac{number\ of\ outcomes\ which\ give\ A}{total\ number\ of\ outcomes} = \frac{r}{n}$$

1.1.1 Random Variables

Random variable: A random variable is a function that has a real number of the outcome of an experiment S. It is represented as

$$X(a) = x_a$$

Where X is the domain of all outcomes.

Single Random Variable

Single Random Variable is one in which it has a single observation.

Q1 Consider the experiment of tossing a coin two times. Let X be the r.v giving the number of heads obtained. (a) What is the range of X? (b) Find the probabilities P(X=0), P(X=1) and P(X=2)

Solution:

Tossing a coin two times, then the sample space is

$$S = \{HH, HT, TH, TT\}$$

(a) The range of X is $R_X = \{0,\ 1,\ 2\}$

Both are heads then $X = 2$, any one head then $X = 1$, (HT or TH) Both are tails then $X = 0$

(X=0)={TT}

$$P(X = 0) = \frac{1}{4}$$

(X=1)={HT or TH}

$$P(X = 1) = \frac{2}{4} = \frac{1}{2}$$

(X=2)={HH}

$$P(X = 2) = \frac{1}{4}$$

Example:2

Consider a sample space of possible outcome when three electronic components are tested is $S = \{NNN, NND, NDN, DNN, NDD, DND, DDN, DDD\}$, where N denotes nondefective and D denotes defective. Consider a random variable X for defective component. X may be be assigned a numerical value of (0, 1, 2, 3).

$X = 0$ when there is defective, $X = 1$ when any one is defective, $X = 2$ when any two are defective, and $X = 3$ when all three are defective.

Q1 Consider the experiment of throwing a fair die. Let X be the r.v which assigns 1 if the number that appears is even and 0 if the number that appears is odd. (a) What is the range of X? (b) Find P(X=1) and P(X=0)

Solution:

The sample space of S on which X is defined is

$$S = \{1, 2, 3, 4, 5, 6\}$$

(a) The range of X is $R_X = \{0,\ 1\}$ where 0 corresponds to if the number that appears is odd and 1 corresponds to if the number that appears is even.

(b)

(X=1)={2,4,6}

$$P(X = 1) = \frac{3}{6} = \frac{1}{2}$$

(X=0)={1,3,5}

$$P(X = 0) = \frac{3}{6} = \frac{1}{2}$$

Types of Random Variables:

There are three types

1. **Discrete random variable:** A random variable that has finite or countable infinite possible values. Example: Tossing a Coin, Throwing Dice.
2. **Continuous random variable:** A random variable that has an (continuous) interval for its set of possible values.
3. **Mixed random variable:** A random variable that has discrete and continuous values.

1.1.2 Cumulative Distribution Function (cdf):

The cumulative distribution function, or cdf, F_X of a random variable X is defined as the probability that, the random variable X takes a value less than or equal to x

$$F_X(x) = P\{a : X(a) \leq x\}$$

Properties Of Discrete CDF F_X:

1. If the independent variable $x = \infty$ i.e.,
$$F_X(\infty) = P\{a : X(a) \leq \infty\} = P(S) = 1$$
2. If the independent variable $x = -\infty$ i.e.,
$$F_X(-\infty) = P\{a : X(a) \leq -\infty\} = P(\phi) = 0$$
which is impossible event.
3. $$0 \leq F_X(x) \leq 1$$
4. Consider a two random variables x_1 and x_2 such that $x_2 > x_1$, then
$$F_X(x_2) - F_X(x_1) =$$
$$= \quad P\{a : X(a) \leq x_2\} - P\{a : X(a) \leq x_1\} \geq 0$$

The cumulative distribution function (cdf) must be monotone nondecreasing i.e, the derivative of cdf must be always be nonnegative.

$$\frac{d}{dx}F_X(x) \quad \geq \quad 0$$

The cumulative distribution function (cdf) is also written as

$$F_X(x) \quad = \quad P\{X \leq x\}$$

The cumulative distribution function (cdf) must be monotone nondecreasing i.e, the derivative of cdf must be always be nonnegative.

$$\frac{d}{dx}F(x) \quad \geq \quad 0$$

1.1.3 Probability Density Function (pdf)

The probability distribution for a random variable describes how probabilities are distributed over the values of the random variable.
If X is continuous random variable then, Probability Density Function of X is defined as

$$f(x) \quad = \quad \frac{d}{dx}F(x)$$

Properties Of pdf:

1. The pdf is always positive
$$f(x) \geq 0 \quad for\ all\ x$$
2. The area under the curve is always 1.
$$\int_{-\infty}^{\infty} f(x)dx = 1$$
3. Consider a two random variables x_1 and x_2 such that $x_2 > x_1$, then
$$\begin{aligned} P\{x_1 < X \leq x_2\} &= F_X(x_2) - F_X(x_1) \\ &= \int_{x_1}^{x_2} f(x)dx \end{aligned}$$

Note:

$$\begin{aligned} P\{a \leq X \leq a\} &= \int_a^a f(x)dx = 0 \\ P\{a \leq X \leq b\} &= P\{a < X < b\} \end{aligned}$$

Probability Distribution Function (pdf)

The probability distribution for a random variable X gives the possible values for X, and the probabilities associated with each possible value.

Probability Mass Function: (pmf)

A Probability P_i associated with a discrete random variable is called a probability mass function pmf f $f(x_i) = P_r(x_i)$ is the probability that X has the value x_i

Properties Of Probability Mass Function :

1. The pmf is always positive
$$0 \leq f(x_i) \leq 1$$
2. The sum of pmfs is .
$$\sum_i f(x_i) = f(x_1) + f(x_2).... = 1$$

Mean, Variance and Standard deviation:

The mean or expected value of a random variable X is:
$$\mu = E(X) = \int_{-\infty}^{\infty} xf(x)dx$$
The variance of a random variable X is:
$$\sigma^2 = Var(V) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = E(x^2) - [E(x)]^2$$
The standard deviation of a random variable X is:
$$\sigma = \sqrt{\sigma^2}$$

Proof

$$\begin{aligned}
E(x-\mu)^2 &= E(x^2 - 2\mu x + \mu^2) \\
&= E(x^2) - 2E(x)\mu + \mu^2 = E(x^2 - 2\mu \times \mu + \mu^2 \\
&= E(x^2) - 2\mu^2 + \mu^2) = E(x^2) - 2\mu^2 \\
&= E(x^2) - [E(x)]^2
\end{aligned}$$

Mean, Variance and Standard deviation for discrete random variable:

The mean or expected value of a random variable X is:

$$\mu = E(X) = \sum_{i=1}^{R} x_i Pr(X = x_i) = \sum_{i=1}^{R} x_i f(x_i)$$

The variance of a random variable X is:

$$\sigma^2 = Var(V) = \sum_{i=1}^{R} (x-\mu)^2 Pr(X = x_i) = \sum_{i=1}^{R} (x-\mu)^2 f(x_i)$$

1.2 Discrete Random Variable

Example 1:

Consider a sample space of possible outcome for tossing a 3 coins at the same time is

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Consider a random variable X for *heads*. X may be be assigned a numerical value of (0, 1, 2, 3).
P(X=0)= TTT=$\frac{1}{8}$, No heads
P(X=1)= HTT, THT,TTH=$\frac{3}{8}$ one head
P(X=2)= HHT,HTH,THH =$\frac{3}{8}$ Two heads
P(X=3)= HHH=$\frac{1}{8}$ Three heads
$F(0) = P(X \leq 0) = P(X = 1) = \frac{1}{8}$
$F(1) = P(X \leq 1) = P(X = 0) + P(X = 1) = \frac{1}{8} + \frac{3}{8} = \frac{4}{8}$
$F(2) = P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} = \frac{7}{8}$
$F(3) = P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = \frac{8}{8} = 1$

$X = x$	0	1	2	3
P(X)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
$F(x) = P(X \leq x)$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{7}{8}$	$\frac{8}{8} = 1$

Example 2:

Two dice are thrown. Let X assign to each point (a,b) in S the maximum number of its number i.e. $X(a, b) = max(a, b)$. Find the probability distribution of the random variable X with $X(S) = \{1, 2, 3, 4, 5, 6\}$

$$\left\{\begin{matrix} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6) \\ (3,1),(3,2),(2,3),(3,4),(3,5),(3,6) \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6) \\ (3,1),(5,2),(5,3),(5,4),(5,5),(5,6) \\ (6,1),(6,2),(6,3),(6,40,(6,5),(6,6) \end{matrix}\right\}$$

X[(1,1)]=max(1,1)=1
P(X=1)=P(1)=$\frac{1}{36}$
X[(2,1),(2,2),(1,2),]2
P(X=2)=P(2)=$\frac{3}{36}$
X[(3,1),(3,2),(3,3),(2,3),(1,3),]=3
P(X=3)=P(3)=$\frac{5}{36}$
X[(4,1),(4,2),(4,3),(4,4),(1,4),(2,4),(3,4)]=4
P(X=4)=P(4)=$\frac{7}{36}$
X[(5,1),(5,2),(5,3),(5,4),(5,5),(1,5),(2,5),(3,5),(4,5)]=5
P(X=5)=P(5)=$\frac{9}{36}$
X[(6,1),(6,2),(6,3),(6,4),(6,5),(6,6),(1,6),(2,6),(3,6),(4,6),(5,6)]=6
P(X=6)=P(6)=$\frac{11}{36}$

$X = x$	1	2	3	4	5	6
P(X)	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

Example 3:

A random variable X takes the values -2,-1,0,1 with probabilities $\frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}$ respectively. Find and draw the probability distribution and cumulative distribution function.

Solution:

$X = x$	-2	-1	0	1
P(X)	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$
$F(x) = P(X \leq x)$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{4}{8}$	$\frac{8}{8} = 1$

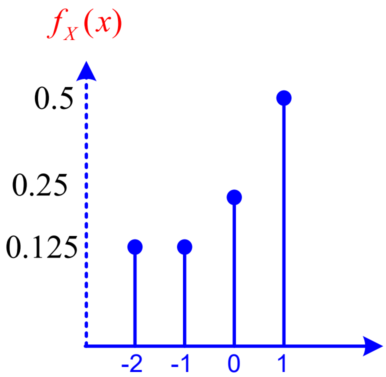

Figure 1.1: probability distribution function

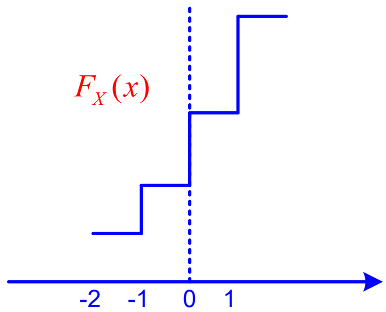

Figure 1.2: Cumulative distribution function

Example 4:

A random variable X has the following distribution. Find the vales of (i) k **(ii)** $P(X \leq 2)$ **(iii)** $P(2 \leq X \leq 5)$

$X = x$	1	2	3	4	5	6	7	8
P(X)	k	$2k$	$3k$	$4k$	$5k$	$6k$	$7k$	$8k$

Solution:

$$\begin{aligned}\sum P(X) = 1 &= k + 2k + 3k + 4k + 5k + 6k + 7k + 8k = 1\\ 36k &= 1\\ k &= \frac{1}{36}\end{aligned}$$

(ii) $P(X \leq 2)$

$$\begin{aligned}P(X \leq 2) &= P(X = 1) + P(X = 2)\\ &= k + 2k = 3k = 3 \times \frac{1}{36} = 0.08\end{aligned}$$

(iii) $P(2 \leq X \leq 5)$

$$\begin{aligned}P(2 \leq X \leq 5) &= +P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)\\ &= 2k + 3k + 4k + 5k = 14k = 14 \times \frac{1}{36} = 0.389\end{aligned}$$

Example 5:

The random variable X has a probability distribution $P(X)$ **of the following form, where** k **is some number.**

$$P(X)\begin{cases} k & if\ \ X = 0\\ 2k & if\ \ X = 1\\ 3k & if\ \ X = 2\\ 0 & otherwise\end{cases}$$

(a) Determine the value of k (b) Find $P(X < 2), P(X \leq 2) and P(X \geq 2)$

Solution:

$X = x$	0	1	2	otherwise
P(X)	k	$2k$	$3k$	0

$$\sum P(X) = 1 = P(0) + P(1) + P(2) + P(otherwise) = k + 2k + 3k + 0 = 6k$$

$$k = \frac{1}{6}$$

(b)
$P(X < 2)$

$$P(X < 2) = P(0) + P(1) = k + 2k = 3k = 3 \times \frac{1}{6} = \frac{1}{2}$$

$P(X \leq 2)$

$$P(X \leq 2) = P(0) + P(1) + P(2) = k + 2k + 3k = 6k = 6 \times \frac{1}{6} = 1$$

$P(X \geq 2)$

$$P(X \geq 2) = P(2) + P(otherwise) = 3k + 0 = 6k = 3 \times \frac{1}{6} = \frac{1}{2}$$

Example 6:

Suppose a discrete r.v X has the following pmfs $P_X(1) = \frac{1}{2}$, $P_X(2) = \frac{1}{4}$, $P_X(3) = \frac{1}{8}$, $P_X(4) = \frac{1}{8}$. (a) Find and sketch the cdf $F_X(x)$ of the r.v X (b) Find (i) $P(X \leq 1)$, (ii) $P(1 < X \leq 3)$, (iii) $P(1 \leq X \leq 3)$

Solution:

(a)

$X = x$	0	1	2	3	4
$P(X = x)$	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$
$F_X(x) = P(X \leq x)$	0	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{7}{8}$	1

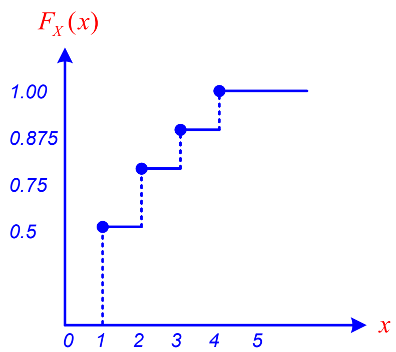

Figure 1.3: Cumulative distribution function (cdf)

(b) Find (i) $P(X \leq 1)$,

$$F_X(x) = P(X \leq 1) = \frac{1}{2}$$

(ii) $P(1 < X \leq 3)$,

$$P(1 < X \leq 3) = F_X(3) - F_X(1) = \frac{7}{8} - \frac{1}{2} = \frac{3}{8}$$

(iii) $P(1 \leq X \leq 3)$

$$P(1 \leq X \leq 3) = P(X = 1) + F_X(3) - F_X(1) = \frac{1}{2} + \frac{7}{8} - \frac{1}{2} = \frac{7}{8}$$

Example 7:

A random variable X has the following distribution. Plot the pdf and cdf of the discrete random variable X

$X = x$	x_a	x_b	x_c
$P(X = x)$	0.24	0.32	0.44
$F(x) = P(X \leq x)$	0.24	0.56	1

Solution:

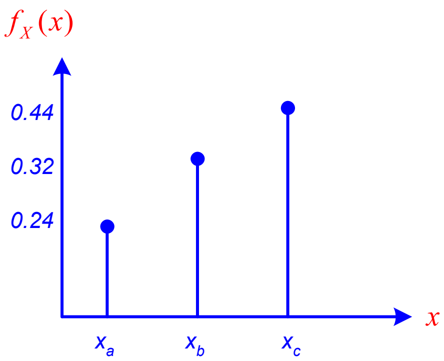

Figure 1.4: Probability density function (pdf)

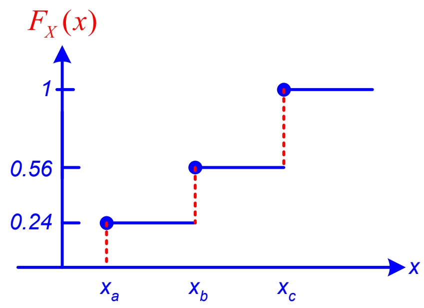

Figure 1.5: Cumulative distribution function (pdf)

Example 8:

A random variable X has the following distribution. Plot the pdf and cdf of the discrete random variable X

$x = i$	0	1	2	3	4	5	6
P_i	0.05	0.15	0.22	0.22	0.17	0.10	0.09
$\sum_{i=0}^{i} P_i$	0.05	0.20	0.42	0.64	0.81	0.91	1.00

Solution:

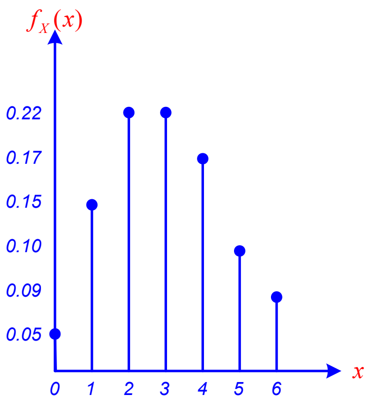

Figure 1.6: Probability density function (pdf)

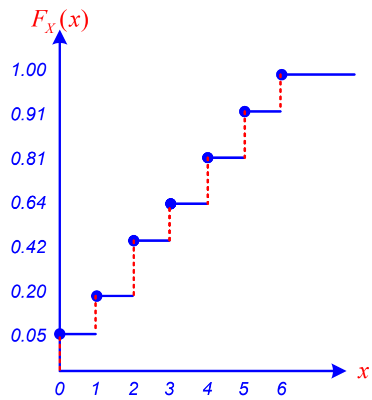

Figure 1.7: Cumulative distribution function (pdf)

Exercise:20 Given the following table,. Plot the pdf and cdf of the discrete random variable X. b) Write the expressions for $f_Y(y)$ and $F_Y(y)$ using unit delta functions and unit step functions

Solution:

$x = i$	1	2	3	4	5
Y_k	2.1	3.2	4.8	5.4	6.9
$P(Y_k)$	0.20	0.21	0.19	0.14	0.26
$F_Y(Y_k)$	0.20	0.41	0.60	0.74	1.00

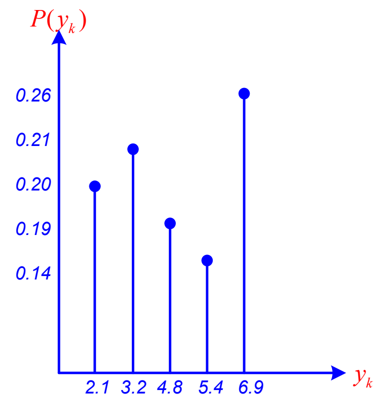

Figure 1.8: Probability density function (pdf)

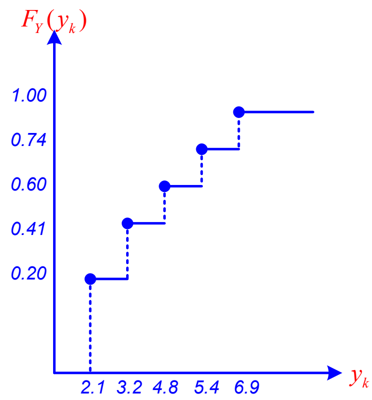

Figure 1.9: Cumulative distribution function (pdf)

$$
\begin{aligned}
f_Y(y) &= 0.20\delta(y-2.1)+0.21\delta(y-3.2)+0.19\delta(y-4.8)+0.14\delta(y-5.4)+0.26\delta(y-6.9) \\
F_Y(y) &= 0.20u(y-2.1)+0.21u(y-3.2)+0.19u(y-4.8)+0.14u(y-5.4)+0.26u(y-6.9)
\end{aligned}
$$

Exercise:21 Given the following table, a) Plot the pdf and cdf of the discrete random variable X. b) Write the expressions for $f_X(x)$ and $F_X(x)$ using unit delta functions and unit step functions

$k = i$	1	2	3	4	5
x_k	2.1	3.2	4.8	5.4	6.9
$P(X_k)$	0.21	0.18	0.20	0.22	0.19
$F_X(X_k)$	0.21	0.39	0.59	0.81	1.00

Solution:

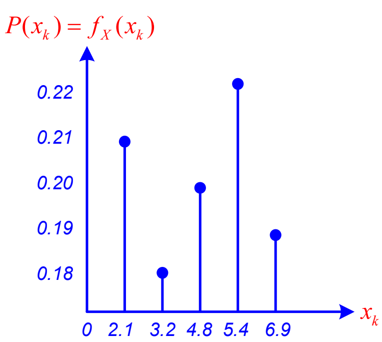

Figure 1.10: Probability density function (pdf)

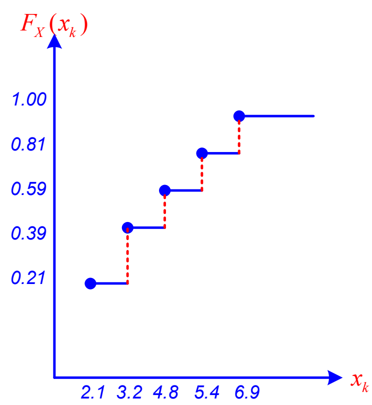

Figure 1.11: Cumulative distribution function (pdf)

$$
\begin{aligned}
f_X(x) &= 0.21\delta(x-2.1)+0.18\delta(x-3.2)+0.20\delta(x-4.8)+0.22\delta(x-5.4)+0.19\delta(x-6.9) \\
F_X(x) &= 0.21u(x-2.1)+0.18u(x-3.2)+0.20u(x-4.8)+0.22u(x-5.4)+0.19u(x-6.9)
\end{aligned}
$$

Exercise:22 Given the following table, a) Plot the pdf and cdf of the discrete random variable Z. b) Write the expressions for $f_Z(z)$ and $F_Z(z)$ using unit delta functions and unit step functions

$k=i$	1	2	3	4	5
z_k	2.1	3.2	4.8	5.4	6.9
$P(Z_k)$	0.19	0.22	0.20	0.18	0.21
$F_Z(Z_k)$	0.19	0.41	0.61	0.79	1.00

Solution:

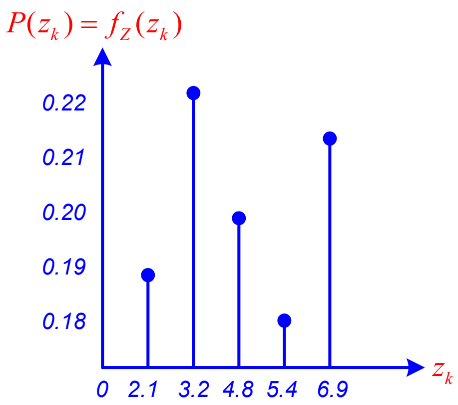

Figure 1.12: Probability density function (pdf)

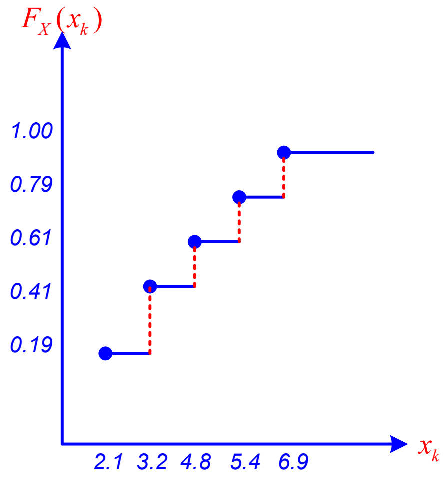

Figure 1.13: Cumulative distribution function (pdf)

$$
\begin{aligned}
f_Z(z) &= 0.19\delta(z-2.1)+0.22\delta(z-3.2)+0.20\delta(z-4.8)+0.18\delta(z-5.4)+0.21\delta(z-6.9) \\
F_Z(z) &= 0.19u(z-2.1)+0.22u(z-3.2)+0.20u(z-4.8)+0.18u(z-5.4)+0.21u(z-6.9)
\end{aligned}
$$

Bernoulli Trial:

Definition: *A Bernoulli trial is a random experiment that can have one of two outcomes which are usually labeled as **success** and **failure.*** Example:

- Tossing coins
- Rolling Dice: The probability of a roll of two die resulting in a double six.

Let X be a Bernoulli random variable with parameter p, where $0 < p < 1$. The probability function $p_X(x)$ of X is given by

$$p_X(x) = p_r(x) = \begin{cases} p & if\ x = 1 \\ 1-p & if\ x = 0 \end{cases}$$

The mean of Bernoulli random variable is

$$\begin{aligned} \mu_X = E[X] &= \sum x p_X(x) \\ &= (1 \times p + 0 \times (1-p) \\ &= p \end{aligned}$$

The variance of X is equal to p(1p), a result obtained as follows:

$$\begin{aligned} var(X) = &= \sum (x-\mu)^2 p_X(x) \\ &= (0-p)^2(1-p)^2 + (0-p)^2 p \\ &= p(1-p)[p + (1-p)] \\ &= p(1-p) \end{aligned}$$

1.3 Binomial Distribution:

Definition: *The random variable X that counts the number of successes, k, in the n trials is said to have a binomial distribution with parameters n and p, written bin(k; n; p).*

$$\begin{aligned} P(X = k) &= \binom{n}{k} p^k (1-p)^{n-k} \qquad 0 \le k \le n \\ f(x) &= \sum_{k=0}^{n} P(X = k)\delta(x-k) \\ F(x) &= \sum_{k=0}^{n} P(X = k)u(x-k) \end{aligned}$$

where $\binom{n}{k}$ counts the number of outcomes that include exactly k successes and n-k failures.

Q1 Consider an experiment in tossing an ideal fair coin with the probability of tossing a head is $p = 0.5$ and the probability of a tail is $q = 1 - p = 1 - 0.5 = 0.5$. What is the probability of tossing exactly 7 heads in 10 tosses.

Solution: $P(X = 7)$

$$\begin{aligned}
P(X=7) &= \begin{pmatrix} n \\ k \end{pmatrix} p^k(1-p)^{n-k} \\
&= \begin{pmatrix} 10 \\ 7 \end{pmatrix} (0.5)^7(1-0.5)^{10-7} \\
&= \frac{10!}{7!(10-7)!}(0.5)^7(0.5)^3 \\
&= 120 \times 0.0078125 \times 0.125 \\
&= 0.11718
\end{aligned}$$

Q2 A biased coin is tossed 6 times. The probability of heads on any toss is 0.3. Let X denote the number of heads that come up. Calculate:

a) $\mathbf{P(X=2)}$

b) $\mathbf{P(X=3)}$

c) $P(1 < X \leq 5)$

Solution:

a) $P(X=2)$

$$\begin{aligned}
P(X=2) &= \begin{pmatrix} n \\ k \end{pmatrix} p^k(1-p)^{n-k} \\
&= \begin{pmatrix} 6 \\ 2 \end{pmatrix} (0.3)^2(1-0.3)^{6-2} \\
&= \frac{6!}{2!(6-2)!}(0.3)^2(0.7)^4 \\
&= \frac{6 \times 5}{2}(0.09)(0.2401) \\
&= 0.324
\end{aligned}$$

b) $P(X=3)$

$$\begin{aligned}
P(X=3) &= \begin{pmatrix} 6 \\ 3 \end{pmatrix} (0.3)^3(0.7)^3 \\
&= \frac{6!}{3!(6-3)!}(0.3)^3(0.7)^3 \\
&= \frac{6 \times 5 \times 4 \times 3 \times 2}{3 \times 2 \times 3 \times 2}(0.027)(0.3431) \\
&= 0.1852
\end{aligned}$$

$$\begin{aligned}
P(X=4) &= \begin{pmatrix} 6 \\ 4 \end{pmatrix} (0.3)^4(0.7)^2 \\
&= 15 \times 0.0081 \times 0.49 \\
&= 0.0595
\end{aligned}$$

$$\begin{aligned}
P(X=5) &= \begin{pmatrix} 6 \\ 5 \end{pmatrix} (0.3)^5(0.7)^1 \\
&= 6 \times 0.00243 \times 0.7 \\
&= 0.0102
\end{aligned}$$

c) $P(1 < X \leq 5)$

$$\begin{aligned} P(1 < X \leq 5) &= P(X=2) + P(X=3) + P(X=4) + P(X=5) \\ &= 0.324 + 0.185 + 0.059 + 0.01 \\ &= 0.578 \end{aligned}$$

Q3. Consider an experiment in tossing ideal fair coin four times with the probability of tossing a head is $p = 0.5$ and the probability of a tail is $q = 1-p = 1-0.5 = 0.5$. If X represents the number heads in the four tosses.then the random variable takes the values { 0,1,2,3,4}. Where $X = 0$ is no head, $X = 1$ one head, $X = 2$ two head and so on. What is the probability of each random variable X.

Solution:

$P(X = 0)$

$$\begin{aligned} P(X=0) &= \binom{n}{k} p^k (1-p)^{n-k} \\ &= \binom{4}{0} (0.5)^0 (1-0.5)^{4-0} \\ &= 1 \times 1 \times \frac{1}{16} \\ &= \frac{1}{16} \\ P(X=1) &= \binom{4}{1} (0.5)^1 (0.5)^3 \\ &= 4 \times \frac{1}{2} \times \frac{1}{8} \\ &= \frac{4}{16} = \frac{1}{4} \\ P(X=2) &= \binom{4}{2} (0.5)^2 (0.5)^2 \\ &= 6 \times 0.25 \times 0.25 \\ &= 0.375 \\ P(X=3) &= \binom{4}{3} (0.5)^3 (0.5)^1 \\ &= 4 \times 0.125 \times 0.5 \\ &= 0.25 \\ P(X=4) &= \binom{4}{4} (0.5)^4 (0.5)^0 \\ &= 1 \times 0.0625 \times 1 \\ &= 0.0625 \end{aligned}$$

Q4. In a hurdle a player has to cross 10 hurdles. The probability that he will clear each hurdle is $5/6$. What is the probability that he will knock down the fewer than 2 hurdles.

Solution:

$p = \frac{5}{6}, q = (1-p) = \frac{1}{6}$

Let X be the random variable that represents the number of times the player will knock down the hurdle. X is a binomial distribution with n=10 $p = \frac{5}{6}$

$$
\begin{aligned}
P(X<0) &= P(X=0)+P(X=1)\\
&= \begin{pmatrix} 10 \\ 0 \end{pmatrix} p^{10}(1-p)^{0} + \begin{pmatrix} 10 \\ 1 \end{pmatrix} p^{9}(1-p)^{1}\\
&= 1\times 1\times \left(\frac{5}{6}\right)^{10} + 10\times\left(\frac{5}{6}\right)^{9}\times\left(\frac{1}{6}\right)\\
&= 0.4845
\end{aligned}
$$

1.4 Poisson Distribution

If X is a random variable which follows a Poisson Distribution with mean λ, then the probability mass function is given by the formula:

$$P_X(k) = \frac{e^{-\lambda}\lambda^k}{k!} \quad k = 0, 1, ...$$

The cdf of X is

$$F_X(x) = e^{-\lambda}\sum_{k=0}^{n}\frac{\lambda^k}{k!} \quad n \le x \le n+1$$

The mean of the poison distribution is

$$\mu = E[X] = \lambda$$

The variance of the poison distribution is

$$\sigma = Var[X] = \lambda$$

Examples of Poisson r.v.'s are

- The number of telephone calls arriving at a switching center during various intervals of time
- The number of customers entering a bank during various intervals of time
- Failures of a machine in one month
- Number of typing errors in a page

Q1 Let X be a Poisson r.v. with parameter with $\lambda = 4$.
(a) Find $P(X > 2)$.

Solution:

(a) Find $P(X > 2)$ with $\lambda = 4$.

$$\begin{aligned} P(X > 2) &= 1 - P(X \le 2) \\ P(X \le 2) &= e^{-\lambda}\sum_{k=0}^{n}\frac{\lambda^k}{k!} = e^{-4}\sum_{k=0}^{2}\frac{4^k}{k!} \\ &= e^{-4}(1 + 4 + 8) = 0.238 \end{aligned}$$

$$P(X > 2) = 1 - P(X \le 2) = 1 - 0.238 = 0.762$$

Q2 Let X be a Poisson distribution such that $P(1) = P(2)$. Find $P(4)$ [?]

Solution:

$$\begin{aligned} P_X(k) &= \frac{e^{-\lambda}\lambda^k}{k!} \quad k = 0, 1, .. \\ P(1) &= P(2) \\ \frac{e^{-\lambda}\lambda^1}{1!} &= \frac{e^{-\lambda}\lambda^2}{2!} \\ \frac{\lambda}{1} &= \frac{\lambda^2}{2} \\ \lambda &= 2 \\ P_X(4) &= \frac{e^{-2}2^4}{4!} = \frac{0.1353 \times 16}{24} = 0.0902 \end{aligned}$$

Q3 Let X be a Poisson distribution with $\lambda = 4$. Find the probability of $0 \le k \le 5$. [?]

Solution:

$$\begin{aligned}
P_X(k) &= \frac{e^{-\lambda}\lambda^k}{k!} \quad k = 0, 1, .. \\
P(0 \le k \le 5) &= P(X = 0) + 0P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) \\
&= \frac{e^{-4}4^0}{0!} + \frac{e^{-4}4^1}{1!} + \frac{e^{-4}4^2}{2!} + \frac{e^{-4}4^3}{3!} + \frac{e^{-4}4^4}{4!} + \frac{e^{-4}4^5}{5!} \\
&= e^{-4}\left[1 + 4 + \frac{4^2}{1.2} + \frac{4^3}{1.2.3} + \frac{4^4}{1.2.3.4} + \frac{4^5}{1.2.3.4.5}\right] \\
&= e^{-4}\left[\frac{600 + 960 + 1280 + 1280 + 1024}{120}\right] \\
&= e^{-4}[42.87] = 0.7851
\end{aligned}$$

Q4 Consider a petrol pump station in which the cars arrival rate is poisson and arrive at an average of 50/hour. If all cars are assumed to require one minute to obtain fuel. What is the probability that a waiting line will occur at the pump?

Solution:

$$\lambda = \frac{50}{60} = 0.83$$

The cars are waiting only when two or more cars arrive at the station. Then required probability is $P(X \ge 2)$

$$\begin{aligned}
P_X(k) &= \frac{e^{-\lambda}\lambda^k}{k!} \quad k = 0, 1, .. \\
P(X \ge 2) &= 1 - P(X < 2) \\
&= 1 - [P(X = 0) + P(X = 1)] \\
&= 1 - \left[\frac{e^{-\lambda}\lambda^0}{0!} + \frac{e^{-\lambda}\lambda^1}{1!}\right] = 1 - \left[\frac{e^{-0.83}0.83^0}{0!} + \frac{e^{-0.83}0.83^1}{1!}\right] \\
&= 1 - [0.4346 + 0.3622] \\
&= 0.2030
\end{aligned}$$

Q5 The probability that an individual suffers a bad reaction due to certain injection is 0.001. Determine the probability that out of 2000 individuals (i)exactly 3 (ii) more than 2 individuals will suffer bad reaction. [?]

Solution:
$p = 0.001$ and $n = 2000$

$$\lambda = np = 0.001 \times 2000 = 2$$

(i)The probability that exactly 3 individuals will suffer bad reaction is

$$\begin{aligned}
P_X(k) &= \frac{e^{-\lambda}\lambda^k}{k!} \quad k = 0, 1, .. \\
P(X = 3) &= \frac{e^{-2}2^3}{3!} \\
&= 0.1804
\end{aligned}$$

(i)The probability that more than 2 individuals will suffer bad reaction is

$$\begin{aligned}
P_X(k) &= \frac{e^{-\lambda}\lambda^k}{k!} \quad k = 0, 1, .. \\
P(X \geq 2) &= 1 - P(X \leq 2) \\
P(X \geq 2) &= 1 - [P(X = 0) + P(X = 1) + P(X = 2)] \\
&= 1 - \left[\frac{e^{-2}2^0}{0!} + \frac{e^{-2}2^1}{1!} + \frac{e^{-2}2^2}{3!}\right] = 1 - e^{-2}\left[1 + 2 + 2\right] \\
&= 1 - [0.6665] \\
&= 0.3333
\end{aligned}$$

Q6 The probability of getting no misprint in a page of a book is e^{-4} . Determine the probability that a page of a book contains more than or equal to 2 misprints. [?]

Solution:

$$P(X = 0) = e^{-4}$$

$$e^{-4} = \frac{e^{-\lambda}\lambda^k}{k!} = \frac{e^{-\lambda}4^0}{0!}$$

$$\lambda = 4$$

(i) Book contains more than or equal to 2 misprints is

$$\begin{aligned}
P_X(k) &= \frac{e^{-\lambda}\lambda^k}{k!} \quad k = 0, 1, .. \\
P(X \geq 2) &= 1 - P(X < 2) \\
&= 1 - [P(X = 0) + P(X = 1)] \\
&= 1 - \left[\frac{e^{-4}4^0}{0!} + \frac{e^{-4}4^1}{1!}\right] = 1 - e^{-4}\left[1 + 4\right] \\
&= 1 - [0.0916] \\
&= 0.9084
\end{aligned}$$

Q7 Assuming that one in 80 births in case of twins. Calculate the the probability of 2 or more births of twins on a day when 30 births occur using (i) Binomial and (ii)Poisson approximate [?]

Solution:
$p = \frac{1}{80} = 0.0125$ $q = 1 - 0.0125 = 0.9875$ and n=30

$$P(X = x) = \binom{n}{x} p^x q^{n-x}$$

(i) The probability of 2 or more births of twins on a day when 30 births occur is using Binomial distribution

$$\begin{aligned}
P(X \geq 2) &= 1 - P(X < 2) \\
P(X \geq 2) &= 1 - [P(X = 0) + P(X = 1)] \\
&= 1 - \left[\binom{30}{0}(0.0125)^0(0.9875)^{30-0} + \binom{30}{1}(0.0125)^1(0.9875)^{30-1}\right] \\
&= 1 - [(0.9875)^29[0.9875 + 30(0.0125)]] \\
&= 1 - 0.9459 = 0.054
\end{aligned}$$

$\lambda = np = 30 \times 0.0125 = 0.375$ Using Poisson distribution

$$\begin{aligned}
P_X(k) &= \frac{e^{-\lambda}\lambda^k}{k!} \quad k = 0, 1, ..\\
P(X \geq 2) &= 1 - P(X < 2)\\
P(X \geq 2) &= 1 - [P(X = 0) + P(X = 1)]\\
&= 1 - \left[\frac{e^{-0.375}0.375^0}{0!} + \frac{e^{-0.375}0.375^1}{1!}\right] = 1 - e^{-0.375}\,[1 + 0.375]\\
&= 1 - [0.945]\\
&= 0.055
\end{aligned}$$

Q8 Passengers arrive at an airport checkout counter at an average rate of 1.5 per minute. Calculate the probabilities that (i) Almost 4 will arrive at a given time and (ii)At least 3 will arrive during an interval of 2 minutes [?]

Solution:
$\lambda = 1.5$ t=1 $\lambda t = 1.5$

$$\begin{aligned}
P_X(k) &= \frac{e^{-\lambda}\lambda^k}{k!} \quad k = 0, 1, ..\\
P(X \leq 4) &= [P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)]\\
&= e^{-1.5}\left[\frac{1.5^0}{0!} + \frac{1.5^1}{1!} + + \frac{1.5^2}{2!} + \frac{1.5^3}{3!} + \frac{1.5^4}{4!}\right]\\
&= e^{-1.5}\left[1 + 1.5 + \frac{(1.5)^2}{2} + \frac{(1.5)^3}{6} + \frac{(1.5)^4}{24}\right]\\
&= e^{-1.5}[2.5 + 1.125 + 0.5625 + 0.2109]\\
&= 0.223 \times 4.3984\\
&= 0.9814
\end{aligned}$$

(i) The probability at least 3 will arrive during an interval of 2 minutes is
$\lambda = 1.5$ t=2 $\lambda t = 1.5 \times 2 = 3$

$$\begin{aligned}
P_X(k) &= \frac{e^{-\lambda}\lambda^k}{k!} \quad k = 0, 1, ..\\
P(X \geq 3) &= 1 - [P(X = \leq 2)] = 1 - [P(X = 0) + P(X = 1) + P(X = 2)]\\
&= 1 - \left[\frac{e^{-3}3^0}{0!} + \frac{e^{-3}3^1}{1!} + + \frac{e^{-3}3^2}{2!}\right]\\
&= 1 - e^{-3}\left[1 + 3 + \frac{9}{2}\right]\\
&= 1 - e^{-3}[8.5]\\
&= 0.5768
\end{aligned}$$

Q9 Suppose that a printer circuit board (PCB) manufacturing company has been averaging 35 errors per month and that the company has 2 PCB errors in a day. Arbitrarily choosing a time unit of one day, the average number of errors per day is 35/30=1.1667. The PCB manufacturing company would like to know (i) The probability of no error in a day and (ii)The probability of 2 PCB errors in a day

Solution:
$\lambda = 1.1667$

(i) The probability of no error in a day is

$$
\begin{aligned}
P_X(k) &= \frac{e^{-\lambda}\lambda^k}{k!} \quad k = 0, 1, .. \\
P(X=0) &= \frac{e^{-1.1667}1.1667^0}{0!} \\
&= 0.31139
\end{aligned}
$$

(ii) The probability of 2 PCB errors in a day is

$$
\begin{aligned}
P_X(k) &= \frac{e^{-\lambda}\lambda^k}{k!} \quad k = 0, 1, .. \\
P(X=2) &= \frac{e^{-1.1667}1.1667^2}{2!} \\
&= 0.2119
\end{aligned}
$$

Q10 A car hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as Poisson variant with mean 1.5. Calculate the proportion of days on which (i) Neither car is used and (ii) Some demand is refused [?]

Solution:

i) Neither car is used is (X=0, No demand)

$$
\begin{aligned}
P_X(k) &= \frac{e^{-\lambda}\lambda^k}{k!} \quad k = 0, 1, .. \\
P(X=0) &= \frac{e^{-1.5}1.5^2}{0!} \\
&= 0.2231
\end{aligned}
$$

(ii) The probability Some demand is refused is when x¿2 (More than two booking)

$$
\begin{aligned}
P_X(k) &= \frac{e^{-\lambda}\lambda^k}{k!} \quad k = 0, 1, .. \\
P(X>2) &= 1 - P(X \leq 2) \\
&= 1 - [P(X=0) + P(X=1) + P(X=2)] \\
&= 1 - \left[\frac{e^{-1.5}1.5^0}{0!} + \frac{e^{-1.5}1.5^1}{1!} + + \frac{e^{-1.5}1.5^2}{2!}\right] \\
&= 1 - e^{-1.5}\left[1 + 1.5 + 1.125\right] \\
&= 1 - 0.8087 \\
&= 0.1916
\end{aligned}
$$

1.5 Probability Models

1.5.1 Continuous Random Variable:

Probability density function (pdf): The probability density function is defined as

$$f(x) = \frac{d}{dx}F(x)$$

when the derivative of the cdf exists. The inverse of cdf is

$$F(x) = \int_{-\infty}^{x} f(x)dx$$

- When the random variable of interest can take any value in an interval, it is called continuous random variable.
- Every continuous random variable has an infinite, uncountable number of possible values.

Probability Distributions: The probability distribution for a random variable X gives the possible values for X, and the probabilities associated with each possible value
Probability Mass Function: Probability mass function $f(x)$ for a discrete random variable X having possible values $x_1, x_2, x_3, ...x_n$
$f(x_i) = P_r(x_i)$ is the probability that X has the value x_i
Properties Of Probability Mass Function:

- The probability of any outcome is between 0 and 1

$$0 \leq f(x_i) \leq 1$$

- The sum of the probabilities of all the outcomes equals 1

$$\sum_i f(x_i) = f(x_1) + f(x_2) + f(x_3)... = 1$$

Properties of Continuous Random Variable:

- $f_X(x) \geq 0$
- $\int_{-\infty}^{\infty} f_X(x) = 1$
- $P(x_1 < X \leq x_2) = \int_{-x_1}^{x_2} f(x) = F_X(x_2) - F_X(x_21)$

Example of Continuous Random Variable:

The following are the major Continuous Probability Distributions

- Uniform Distribution
- Exponential Distribution
- The Normal Gaussian Distribution

Q2 The cdf for the random variable X is.

$$F_X(x) = \begin{cases} 1 - \exp\left(\frac{-x^2}{4}\right) & x \geq 0 \\ 0 & otherwise \end{cases}$$

Evaluate $P(1.1 < X < 2.1)$ [?]

Solution:

$$\begin{aligned} P(1.1 < X < 2.1) &= F_X(2.1) - F_X(1.1) \\ &= \left[1 - \exp\left(\frac{-x^2}{4}\right)\right] - \left[1 - \exp\left(\frac{-x^2}{4}\right)\right] \\ &= \exp\left(\frac{-(1.1)^2}{4}\right) - \exp\left(\frac{-(2.1)^2}{4}\right) \\ &= 0.4069 \end{aligned}$$

Q3 The cdf for the random variable Y is.

$$F_Y(y) = \begin{cases} 1 - \exp\left(-0.4y^{0.5}\right) & y \geq 0 \\ 0 & otherwise \end{cases}$$

Evaluate $P(2.5 < Y < 6.2)$ [?]

Solution:

$$\begin{aligned} P(2.5 < Y < 6.2) &= F_Y(6.2) - F_Y(2.5) \\ &= \left[1 - \exp\left(-0.4y^{0.5}\right)\right] - \left[1 - \exp\left(-0.4y^{0.5}\right)\right] \\ &= \exp\left(-0.4(2.5)^{0.5}\right) - \exp\left(-0.4(6.2)^{0.5}\right) \\ &= 0.1619 \end{aligned}$$

Q4 The cdf for the random variable Z is.

$$F_Z(z) = \begin{cases} 1 - \exp\left(-2z^{3/2}\right) & z \geq 0 \\ 0 & otherwise \end{cases}$$

Evaluate $P(0.5 < Z < 0.9)$ [?]

Solution:

$$\begin{aligned} P(0.5 < Z < 0.9) &= F_Z(0.9) - F_Z(0.5) \\ &= \left[1 - \exp\left(-2z^{3/2}\right)\right] - \left[1 - \exp\left(-2z^{3/2}\right)\right] \\ &= \exp\left(-2(0.5)^{3/2}\right) - \exp\left(-2(0.9)^{3/2}\right) \\ &= 0.3118 \end{aligned}$$

Example 1 A continuous random variable has the pdf for the random variable X is.

$$f(x) = \begin{cases} ke^{-x} & if \quad x > 0 \\ 0 & otherwise \end{cases}$$

Determine the constant k [?]

Solution:

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)dx &= 1 \\ \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx &= 1 \\ 0 + \int_{0}^{\infty} ke^{-x}dx &= 1 \\ k\left[-e^{-x}\right]_0^{\infty} &= 1 \\ -k[0-1] &= 1 \\ k &= 1 \end{aligned}$$

Example 2 Is $f(x)$

$$f(x) = \begin{cases} \frac{1}{2}(x+1) & for \quad -1 < x < 1 \\ 0 & otherwise \end{cases}$$

represents the density of random variable has the pdf for the random variable X? [?]

Solution:

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)dx &= \int_{-1}^{1} f(x)dx \\ &= \int_{-1}^{1} \frac{1}{2}(x+1)dx \\ &= \frac{1}{2}\left[\frac{x^2}{2} + x\right]_{-1}^{1} \\ &= \frac{1}{2}\left[\frac{1^2}{2} + 1\right] - \left[\frac{(-1)^2}{2} - 1\right] \\ &= \frac{1}{2}\left[\frac{3}{2} - \frac{-1}{2}\right] \\ &= 1 \end{aligned}$$

Hence

$$f(x) = \begin{cases} \frac{1}{2}(x+1) & for \quad -1 < x < 1 \\ 0 & otherwise \end{cases}$$

is a density function

Example 3 If $f(x)$ is a probability density function defied as

$$f(x) = \begin{cases} kx^3 & for \quad 0 \le x \le 3 \\ 0 & otherwise \end{cases}$$

Find the value of k and find the probability between $x = \frac{1}{2}$ and $x = \frac{3}{2}$[?]

Solution:

If it is a density function, then it must satisfies

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)dx &= 1 \\ \int_{0}^{3} kx^3dx &= 1 \\ k\left[\frac{x^4}{4}\right]_0^3 &= 1 \\ k\left[\frac{3^4}{4} - 0\right] &= 1 \\ \frac{81}{4}k &= 1 \\ k &= \frac{4}{81} \end{aligned}$$

Hence

$$f(x) = \begin{cases} \frac{4}{81}x^3 & for \quad 0 \le x \le 3 \\ 0 & otherwise \end{cases}$$

$$P\left(\frac{1}{2} \le X \le \frac{3}{2}\right)$$

$$\begin{aligned} P\left(\frac{1}{2} \le X \le \frac{3}{2}\right) &= \int_{\frac{1}{2}}^{\frac{3}{2}} f(x)dx \\ &= k\int_{\frac{1}{2}}^{\frac{3}{2}} f(x)x^3dx \\ &= k\left[\frac{x^4}{4}\right]_{\frac{1}{2}}^{\frac{3}{2}} \\ &= \frac{k}{4}\left[\left(\frac{3}{2}\right)^4 - \left(\frac{1}{2}\right)^4\right] \\ &= \frac{4}{81}\frac{1}{4}\left[\frac{80}{16}\right] \\ &= 0.0617 \end{aligned}$$

Q 14 A pdf is described by $c(x-6)$ for all values of X between 6 and 10 is 0 otherwise. Find the value that c must have, and evaluated $P\{X > 8.0\}$ [?]

Solution:

If it is a density function, then it must satisfies

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)dx &= 1\\
\int_{6}^{10} c(x-6)dx &= 1\\
c\left[\frac{x^2}{2}-6x\right]_6^{10} &= 1\\
c\left[\left(\frac{10^2}{2}-6\times 10\right)-\left(\frac{6^2}{2}-6\times 6\right)\right] &= 1\\
c[50-60-18+36] &= 1\\
c &= \frac{1}{8}
\end{aligned}$$

$P\{X > 8.0\}$

$$P\{X > 8.0\} = 1 - P\{X = 8.0\}$$

$$\begin{aligned}
P\{X = 8.0\} &= \int_{6}^{8} c(x-6)dx\\
&= c\left[\frac{x^2}{2}-6x\right]_6^{8}\\
&= c\left[\left(\frac{8^2}{2}-6\times 8\right)-\left(\frac{6^2}{2}-6\times 6\right)\right]\\
&= c[32-48-18+36]\\
&= \frac{1}{8}[2] = 0.25
\end{aligned}$$

$$\begin{aligned}
P\{X > 8.0\} &= 1-0.25\\
&= 0.75
\end{aligned}$$

Q 15 A pdf is described by $c(y-3)$ for all values of Y between 6 and 10 is 0 otherwise. Find the value that c must have, and evaluated $P\{Y > 7.0\}$ [?]

Solution:

If it is a density function, then it must satisfies

$$\int_{-\infty}^{\infty} f(y)dy = 1$$

$$\begin{aligned}
\int_{-\infty}^{\infty} f(y)dy &= 1\\
\int_{6}^{10} c(y-3)dy &= 1\\
c\left[\frac{y^2}{2}-3y\right]_6^{10} &= 1\\
c\left[\left(\frac{10^2}{2}-3\times 10\right)-\left(\frac{6^2}{2}-3\times 6\right)\right] &= 1\\
c[50-30-18+18] &= 1\\
c &= \frac{1}{20}
\end{aligned}$$

$P\{Y > 7.0\}$

$$P\{Y > 7.0\} = 1 - P\{Y = 7.0\}$$

$$\begin{aligned}
P\{Y = 7.0\} &= \int_{6}^{7} c(y-3)dy\\
&= c\left[\frac{y^2}{2}-3y\right]_6^{7}\\
&= c\left[\left(\frac{7^2}{2}-3\times 7\right)-\left(\frac{6^2}{2}-3\times 6\right)\right]\\
&= c[24.5-21-18+18]\\
&= \frac{1}{20}[2] = 0.175
\end{aligned}$$

$$\begin{aligned}
P\{Y > 7.0\} &= 1-0.175\\
&= 0.825
\end{aligned}$$

Q 16 A pdf is described by $c(z-4)$ for all values of Z between 6 and 10 is 0 otherwise. Find the value that c must have, and evaluated $P\{Z > 9.0\}$ [?]

Solution:

If it is a density function, then it must satisfies

$$\int_{-\infty}^{\infty} f(z)dz = 1$$

$$\begin{aligned}
\int_{-\infty}^{\infty} f(z)dz &= 1\\
\int_{6}^{10} c(z-4)dz &= 1\\
c\left[\frac{z^2}{2}-3y\right]_6^{10} &= 1\\
c\left[\left(\frac{10^2}{2}-4\times 10\right)-\left(\frac{6^2}{2}-4\times 6\right)\right] &= 1\\
c[50-40-18+24] &= 1\\
c &= \frac{1}{16}
\end{aligned}$$

$P\{z > 9.0\}$

$$P\{Z > 9.0\} = 1 - P\{Z = 9.0\}$$

$$\begin{aligned}
P\{Z = 9.0\} &= \int_{6}^{9} c(z-4)dz\\
&= c\left[\frac{z^2}{2}-4z\right]_6^{9}\\
&= c\left[\left(\frac{9^2}{2}-4\times 9\right)-\left(\frac{6^2}{2}-4\times 6\right)\right]\\
&= c[40.5-36-18+24]\\
&= \frac{1}{10.5}[16] = 0.65625
\end{aligned}$$

$$\begin{aligned} P\{Z > 9.0\} &= 1 - 0.65625 \\ &= 0.34375 \end{aligned}$$

Q 17 The following is the pdf for the random variable U. Find the value that c must have, and evaluate $F_U(0.5)$ [?]

$$f_U(u) \begin{cases} c\exp(-u/2) & 0 \leq u \leq 1 \\ 0 & otherwise \end{cases}$$

Solution:

If it is a density function, then it must satisfies

$$\begin{aligned} F_U(u) &= \int_0^u ce^{(-u/2)} du \\ 1 &= -2c \left[e^{(-u/2)} \right]_0^1 \\ &= -2c \left[e^{(-1/2)} - e^{(-0)} \right] \\ &= 2c \left[1 - e^{(-1/2)} \right] \\ &= 2c[1 - 0.6065] \\ 1 &= 2c \times 0.3935 = 0.787c \\ c &= \frac{1}{0.787} = 1.2707 \end{aligned}$$

$$\begin{aligned} F_U(u) &= 2c \left[1 - e^{(-u/2)} \right] \\ F_U(0.5) &= 2(1.2707) \left[1 - e^{(-0.5/2)} \right] \\ &= 2.5414 \times [1 - 0.7788] \\ &= 2.5414 \times 0.2212 \\ &= 0.5622 \end{aligned}$$

Q 18 The following is the pdf for the random variable U. Find the value that c must have, and evaluate $F_U(0.5)$ [?]

$$f_U(u) \begin{cases} c\exp(-u/3) & 0 \leq u \leq 1 \\ 0 & otherwise \end{cases}$$

Solution:

If it is a density function, then it must satisfies

$$\begin{aligned} F_U(u) &= \int_0^u ce^{(-u/3)} du \\ 1 &= -3c \left[e^{(-u/3)} \right]_0^1 \\ &= -3c \left[e^{(-1/3)} - e^{(-0)} \right] \\ &= 3c \left[1 - e^{(-1/3)} \right] \\ &= 3c[1 - 0.7165] \\ 1 &= 3c \times 0.2835 = 0.8505c \\ c &= \frac{1}{0.787} = 1.1757 \end{aligned}$$

$$\begin{aligned} F_U(u) &= 3c \left[1 - e^{(-u/3)} \right] \\ F_U(0.5) &= 3(1.1757) \left[1 - e^{(-0.5/3)} \right] \\ &= 2.5414 \times [1 - 0.8464] \\ &= 3.527 \times 0.1536 \\ &= 0.5417 \end{aligned}$$

Q 19 The following is the pdf for the random variable U. Find the value that c must have, and evaluate $F_U(0.5)$ [?]

$$f_U(u) \begin{cases} c\exp(-u/4) & 0 \leq u \leq 1 \\ 0 & otherwise \end{cases}$$

Solution:

If it is a density function, then it must satisfies

$$\begin{aligned} F_U(u) &= \int_0^u ce^{(-u/4)} du \\ 1 &= -4c \left[e^{(-u/4)} \right]_0^1 \\ &= -4c \left[e^{(-1/4)} - e^{(-0)} \right] \\ &= 4c \left[1 - e^{(-1/4)} \right] \\ &= 4c[1 - 0.7788] \\ 1 &= 4c \times 0.2212 = 0.8848c \\ c &= \frac{1}{0.787} = 1.1302 \end{aligned}$$

$$\begin{aligned} F_U(u) &= 4c \left[1 - e^{(-u/3)} \right] \\ F_U(0.5) &= 4(1.1302) \left[1 - e^{(-0.5/4)} \right] \\ &= 4.5208 \times [1 - 0.8825] \\ &= 4.5208 \times 0.1175 \\ &= 0.5312 \end{aligned}$$

1.6 Continuous Uniform Distribution:

Learning Outcomes

On completion, students are able to

- understand the uniform distribution
- understand probability density function (pdf) and Cumulative Distribution Function (cdf)
- calculate the mean and variance of a uniform distribution

The uniform distribution (continuous) is one of the simplest probability distributions in statistics. It is a continuous distribution, this means that it takes values within a specified range, e.g. between a and b.

The **probability density function (pdf)** for a uniform distribution taking values in the range a to b is:

$$f(x) = \begin{cases} k & a < x \le b \\ 0 & otherwise \end{cases}$$

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x \le b \\ 0 & otherwise \end{cases}$$

Proof:

$$\begin{aligned} \int_a^b f(x)dx &= \int_a^b kdx = k[x]_a^b = 1 \\ k(b-a) &= 1 \\ k &= \frac{1}{b-a} \end{aligned}$$

Cumulative Distribution Function (cdf):

$$\begin{aligned} F(x) &= p(X \le x) = \int_a^x f(x)dW \\ &= \int_a^x \frac{1}{b-a}dW \\ &= \frac{1}{b-a}[W]_a^x = \frac{1}{b-a}[x-a] \\ &= \frac{x-a}{b-a} \end{aligned}$$

$$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a < x \le b \\ 1 & x \ge b \end{cases}$$

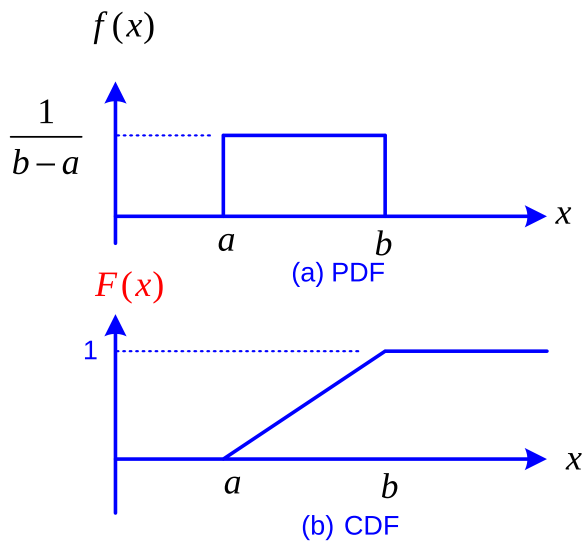

Figure 1.14: Uniform Distribution

Variance:

$$\begin{aligned} Var[X] &= E[X^2] - [E(X)]^2 \\ &= E[X^2] - \left[\frac{b-a}{2}\right]^2 \end{aligned}$$

$$\begin{aligned} E[X^2] &= \int_a^b x^2 \frac{1}{b-a}dx \\ &= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b = \frac{b^3-a^3}{3(b-a)} \\ &= \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} \\ &= \frac{(b^2+ab+a^2)}{3} \end{aligned}$$

$$\begin{aligned} Var[X] &= E[X^2] - \left[\frac{b-a}{2}\right]^2 \\ &= \frac{(b^2+ab+a^2)}{3} - \left[\frac{a^2+b^2+2ab}{4}\right] \\ &= \frac{4b^2+4ab+4a^2-3a^2-3b^2-6ab}{12} \\ &= \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12} \end{aligned}$$

Mean: μ

$$\begin{aligned} E[X] &= \int_a^b xf(x)dx = \int_a^b x\frac{1}{b-a}dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b \\ &= \frac{b^2-a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)} = \frac{a+b}{2} \end{aligned}$$

Standard Deviation σ

$$\begin{aligned} \sigma &= \sqrt{Var[X]} = \sqrt{\frac{(b-a)^2}{12}} \\ &= \frac{b-a}{\sqrt{12}} \end{aligned}$$

Q1 The amount of time, that a person must wait for a bus is uniformly distributed between zero and 15 minutes, inclusive.

a) What is the probability that a person waits fewer than 12.5 minutes?

b) On the average, how long must a person wait? Find the mean, μ, and the standard deviation, σ.

c) Ninety percent of the time, the time a person must wait falls below what value?

Solution:

a). Let X= the number of minutes a person must wait for a bus. $a = 0$ and $b = 15$. $X\ U(0, 15)$. The probability density function is

$$\begin{aligned} f(x) &= \frac{1}{b-a} \\ &= \frac{1}{15-0} \quad for\ 0 \leq x \leq 15 \\ &= \frac{1}{15} = 0.0667 \end{aligned}$$

$$\begin{aligned} P(x < 12.5) &= \frac{x-a}{b-a} = \frac{12.5-0}{15-0} \\ = 0.8333 \end{aligned}$$

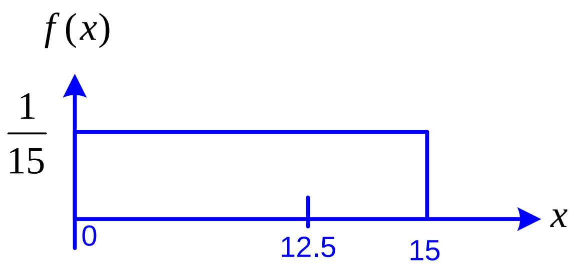

Figure 1.15: Uniform Distribution

b) On average, a must a person wait i.e., mean, μ is

$$\mu = \frac{a+b}{2} = \frac{0+15}{2} = 7.5$$

The standard deviation, σ is

$$\sigma = \sqrt{\frac{(b-a)^2}{12}} = \sqrt{\frac{(12-0)^2}{12}} = 4.3$$

c). Ninety percent of the time, the time a person must wait falls below is

$$\begin{aligned} P(x < k) &= \frac{k-a}{b-a} = \frac{k-0}{15} \\ 0.90 &= \frac{k}{15} \\ k &= 0.90 \times 15 \\ &= 13.5 \end{aligned}$$

The Ninety percent of the time, a person must wait at most 13.5 minutes.

Q2 The amount of time, that a person must wait for a bus is uniformly distributed between zero and 10 minutes.

a) Write the probability density function.

b) Find the probability that a person has to wait between four and six minutes for a bus.

c) Find the probability that a person has to wait between three and seven minutes for a bus

d) Find the probability that a person has to wait between zero and ten minutes for a bus.

Solution:

a). Let X= the number of minutes a person must wait for a bus. a=0 and b=10. X U(0,10). Write the probability density function.

$$\begin{aligned} f(x) &= \frac{1}{10-0} \quad for\ 0 \leq x \leq 10 \\ &= \frac{1}{10} = 0.1 \end{aligned}$$

b) The probability that a person has to wait between four and six minutes for a bus.

$$\begin{aligned} P(4 < x < 6) &= \frac{x-a}{b-a} = \frac{6-4}{10-0} \\ &= 0.2 \end{aligned}$$

c) The probability that a person has to wait between three and seven minutes for a bus.

$$\begin{aligned} P(3 < x < 7) &= \frac{x-a}{b-a} = \frac{7-3}{10-0} \\ &= 0.4 \end{aligned}$$

d) The probability that a person has to wait between zero and ten minutes for a bus.

$$\begin{aligned} P(0 < x < 10) &= \frac{x-a}{b-a} = \frac{10-0}{10-0} \\ &= 1 \end{aligned}$$

Q3 In an apartment a person is taking an elevator to reach his floor. Once he called the elevator, it will take between 0 and 40 seconds and assuming that the elevator arrives uniformly between 0 and 40 seconds after pressing the button.

a) Write down the formula for the probability density function $f(x)$ of the random variable X representing the arrival time.

b) Calculate the probability that elevator takes less than 15 seconds to arrive.

c) Calculate the mean and variance of the distribution and find the cumulative distribution function $F(x)$.

Solution:

a) The interval is [0, 40] the probability density function $f(x)$ is given by

$$\begin{aligned} f(x) &= \frac{1}{40-0} \quad for\ 0 < x \leq 40 \\ &= \frac{1}{40} = 0.025 \end{aligned}$$

b) The probability that elevator takes less than 15 seconds to arrive is

$$\begin{aligned} P(0 \leq x \leq 15) &= \frac{x-a}{b-a} = \frac{15-0}{40-0} \\ &= 0.375 \end{aligned}$$

c)

$$Mean\ E(x) = \frac{40-0}{2} = 20 seconds$$

$$Variance\ \ V(x) = \frac{(40-0)^2}{12} = 33.33$$

Cumulative Distribution Function (cdf)

$$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a < x \leq b \\ 1 & x \geq b \end{cases}$$

$$F(x) = \begin{cases} 0 & x < a \\ \frac{x}{40} & 0 \leq x \leq 40 \\ 1 & x > 40 \end{cases}$$

Q4. The total duration of baseball games are uniformly distributed between 447 hours and 521 hours inclusive.

a) Find a and b and describe what they represent.

b) Write the distribution.

c) Find the mean and the standard deviation.

d) What is the probability that the duration of games for a team between 480 and 500 hours?

e) What is the 65th percentile for the duration of games for a team?

Solution:

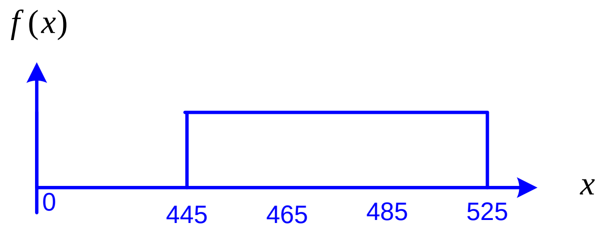

Figure 1.16: Uniform Distribution

a) $a = 447$ and $b = 521$

b) Write the distribution. $X \sim U(447, 521)$

$$\begin{aligned} f(x) &= \frac{1}{521-447} \quad for\ 447 < x \leq 521 \\ &= \frac{1}{74} = 0.0135 \end{aligned}$$

c) The mean and the standard deviation is

$$\mu = \frac{a+b}{2} = \frac{447+521}{2} = 484$$

$$\sigma = \sqrt{\frac{(b-a)^2}{12}} = \sqrt{\frac{(521-447)^2}{12}} = 21.36$$

d) The probability that the duration of games for a team between 480 and 500 hours is

$$\begin{aligned} P(480 < x < 500) &= \frac{x-a}{b-a} = \frac{500-480}{521-447} \\ &= \frac{20}{74} = 0.2703 \end{aligned}$$

e) The 65th percentile for the duration of games for a team is

$$\begin{aligned} 0.65 &= \frac{x-a}{b-a} = \frac{x-480}{521-447} \\ &= \frac{x-447}{74} \\ x - 447 &= 0.65 \times 74 = 48.1 \\ x &= 495.1 \ \ hours \end{aligned}$$

Q5 The amount of time a service technician needs to change the oil in a car is uniformly distributed between 11 and 21 minutes. Let X = the time needed to change the oil on a car.

a) Write the random variable X in words.

b) Write the distribution.

c) Find $P(x > 19)$.

d) Find the 50th percentile

Solution:

a) The random variable X represents the time needed to change the oil in a car.
b) The distribution is. $X \sim U(11, 21)$

$$\begin{aligned} f(x) &= \frac{1}{21-11} \quad for\ 11 < x \leq 21 \\ &= \frac{1}{10} = 0.1 \end{aligned}$$

c) The value of $P(x > 19)$ is.

$$\begin{aligned} P(x > 19) &= 1 - P(x < 19) \\ &= 1 - \frac{19-11}{21-11} = 1 - 0.8 \\ &= 0.2 \end{aligned}$$

e) The 50th percentile is

$$\begin{aligned} 0.5 &= \frac{x-a}{b-a} = \frac{x-11}{21-11} \\ &= \frac{x-11}{10} \\ x - 11 &= 0.5 \times 10 = 5 \\ x &= 16 \ \ minutes. \end{aligned}$$

Q6 The current (in mA) measured in a piece of copper wire is known to follow a uniform distribution over the interval [0, 25]. Write down the formula for the probability density function $f(x)$ of the random variable X representing the current. Calculate the mean and variance of the distribution and find the cumulative distribution function $F(x)$. [?]

Solution:

The probability density function $f(x)$ in the interval [0, 25] is

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x \leq b \\ 0 & otherwise \end{cases}$$

$$f(x) = \begin{cases} \frac{1}{25-0} = 0.04 & 0 < x \leq 25 \\ 0 & otherwise \end{cases}$$

$$E(x) = \frac{25-0}{2} = 12.5mA$$

$$Variance \ \ V(x) = \frac{(25-0)^2}{12} = 52.08mA^2$$

Cumulative Distribution Function (cdf)

$$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a < x \leq b \\ 1 & x \geq b \end{cases}$$

$$F(x) = \begin{cases} 0 & x < a \\ \frac{x}{25} & 0 \leq x \leq 25 \\ 1 & x > 25 \end{cases}$$

Q7 The thickness x of a protective coating applied to a conductor designed to work in corrosive conditions follows a uniform distribution over the interval [20, 40] microns. Find the mean, standard deviation and cumulative distribution function of the thickness of the protective coating. Find also the probability that the coating is less than 35 microns thick. [?]

Solution:

The probability density function $f(x)$ in the interval [20, 40] is

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x \leq b \\ 0 & otherwise \end{cases}$$

$$f(x) = \begin{cases} \frac{1}{40-20} = 0.05 & 20 < x \leq 40 \\ 0 & otherwise \end{cases}$$

The mean is

$$E(x) = \frac{40-20}{2} = 10\mu m$$

The standard deviation is

$$Variance \ \ V(x)\ \sigma^2 = \frac{(40-20)^2}{12} = 33.333$$

$$Standard\ deviation\ \sigma = \sqrt{33.333} = 5.77\mu m$$

Cumulative Distribution Function (cdf)

$$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a < x \leq b \\ 1 & x \geq b \end{cases}$$

$$F(x) = \begin{cases} 0 & x < 20 \\ \frac{x-20}{20} & 20 \leq x \leq 40 \\ 1 & x > 40 \end{cases}$$

The probability that the coating is less than 35 microns thick is

$$\begin{aligned} P(x < 35) &= \frac{35-20}{40-20} \\ &= 0.75 \end{aligned}$$

Q71. The random variable X is uniformly distributed between 0 and 4. The random variable Y is obtained from X using $y = (x-2)^2$. What are the cdf and pdf for y.

Solution:

$$f(x) = \begin{cases} \frac{1}{4-0} = 0.25 \;\; 0 < x \leq 4 \\ \\ 0 \quad otherwise \end{cases}$$

$$F(x) = 0.25x \;\; 0 < x \leq 4$$

$$\begin{aligned} y &= (x-2)^2 \\ (x-2) &= \pm\sqrt{y} \\ x &= \pm\sqrt{y} + 2 \end{aligned} \tag{1.1}$$

$$\begin{aligned} F(y) &= P\{Y \leq y\} \\ &= P\{-Y \leq y\} \\ x &= \pm\sqrt{y} + 2 \end{aligned}$$

Q72. Resistors R are uniformly distributed in the interval $50 \pm 5\Omega$. From circuit theory the conductance $G = \frac{1}{r}$. What are the cdf and pdf for G. [?]

Solution:

$$f(x) = \begin{cases} \frac{1}{55-45} = 0.1 \;\; 45 < r \leq 55 \\ \\ 0 \quad otherwise \end{cases}$$

$$F(x) = \frac{r-45}{55-45} = 0.1(r-45) \;\; 45 < r \leq 55$$

$$\begin{aligned} F(g) &= P\{G \leq g\} \\ &= P\{R \geq 1/r\} \\ &= 1 - F(1/g) \\ &= 1 - 0.10(1/g - 45) \;\; 1/55 < r \leq 1/45 \end{aligned}$$

$$\begin{aligned} f(g) &= \frac{d}{dg}F(g) \\ &= -0.10(-1/g^2) \;\; 1/55 < r \leq 1/45 \end{aligned}$$

Q75. The random variable X is uniformly distributed between 0 and 2 and $y = 2x^3$. What is the pdf for Y. [?]

Solution:

$$y = 3x^3$$

$$x = (y/3)^{1/3}$$

$$f(x) = \begin{cases} \frac{1}{2-0} = 0.5 \;\; 0 < x < 2 \\ \\ 0 \quad otherwise \end{cases}$$

$$F(x) = \frac{x-0}{2-0} = 0.1x \;\; 0 < x < 2$$

$$\begin{aligned} F(y) &= P\{Y \leq y\} \\ &= P\{X \leq (y/3)^{1/3}\} \\ &= F(x)((y/3)^{1/3}) \\ &= 0.5(y/3)^{1/3} \;\; 0 < y < 24 \end{aligned}$$

$$\begin{aligned} f(y) &= \frac{d}{dy}F(y) \\ &= \frac{1}{2(3)^{4/3}y^{2/3}} \;\; 0 < y < 24 \end{aligned}$$

1.7 Exponential Distribution:[?, ?, ?]

Exponential Distribution:

Learning Outcomes
On completion the students are able to

- understand the probability density function of an exponential distribution.
- understand the cumulative distribution function of an exponential distribution.
- understand the mean and variance of an exponential distribution
- use the exponential distribution to solve simple practical problems.
- The exponential distribution is widely used in the field of reliability.
- Reliability deals with the amount of time a product lasts.

Consider a random variable X which is exponentially distributed with parameter λ (failure rate/the process rate) then its probability distribution is given by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ \\ 0 & x < 0 \end{cases}$$

The area under the curve is

$$\begin{aligned} &= \int_{-\infty}^{\infty} f_X(x)dx = \int_0^{\infty} \lambda e^{-\lambda x} dx \\ &= \lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_0^{\infty} \\ &= \frac{\lambda}{-\lambda}[0-1] \\ &= 1 \end{aligned}$$

Cumulative distribution function (cdf)

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ \\ 0 & x < 0 \end{cases}$$

where λ is a rate constant and $\lambda > 0$

Proof

$$\begin{aligned} F(x) &= \int_{-\infty}^{-\infty} f_X(x)dx = \int_0^t \lambda e^{-\lambda x} dx \\ &= \lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_0^t \\ &= \frac{\lambda}{-\lambda}[e^{-\lambda}t - 1] \\ &= 1 - e^{-\lambda}t \end{aligned}$$

Expected Value (Mean)

$$\begin{aligned} E[X] &= \int_{-\infty}^{-\infty} x f_X(x) dx \\ &= \lambda \int_0^{\infty} x e^{-\lambda x} dx \\ &= \lambda \left[x \int_0^{\infty} e^{-\lambda x} dx \right] \\ &- \lambda \left[\int \frac{d}{dx} x \int_0^{\infty} e^{-\lambda x} dx \right] \end{aligned}$$

$$\begin{aligned} &= \lambda \left\{ \left[\frac{x e^{-\lambda x}}{-\lambda} \right]_0^{\infty} - \left[\int_0^{\infty} \frac{e^{-\lambda x}}{-\lambda} dx \right] \right\} \\ &= [0-0] + \left[\frac{e^{-\lambda x}}{-\lambda} \right]_0^{\infty} \\ &= -\frac{1}{\lambda}[0-1] = \frac{1}{\lambda} \end{aligned}$$

Variance

$$\begin{aligned} Var[X] &= E[X^2] - E[X]^2 \\ &= \int_{-\infty}^{-\infty} x^2 f_X(x) dx - \frac{1}{\lambda^2} \end{aligned}$$

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{-\infty} x^2 f_X(x) dx = \lambda \int_0^{\infty} x^2 e^{-\lambda x} dx \\ &= \lambda \left[x^2 \int_0^{\infty} e^{-\lambda x} dx \right] \\ &- \left[\int \frac{d}{dx} x^2 \int_0^{\infty} e^{-\lambda x} dx dx \right] \\ &= \lambda \left[x^2 \frac{e^{-\lambda x}}{-\lambda} \right]_0^{\infty} - \left[\int_0^{\infty} 2x \frac{e^{-\lambda x}}{-\lambda} dx \right] \\ &= [0-0] + \frac{2}{\lambda} \left[\int_0^{\infty} x e^{-\lambda x} dx \right] \\ &= \frac{2}{\lambda} * \frac{1}{\lambda} = \frac{2}{\lambda^2} \end{aligned}$$

$$\begin{aligned} Var[X] &= E[X^2] - E[X]^2 \\ &= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} \\ &= \frac{1}{\lambda^2} \end{aligned}$$

The Standard Deviation of the distribution

$$\begin{aligned} \sigma &= \sqrt{Var[X]} = \sqrt{\frac{1}{\lambda^2}} \\ &= \frac{1}{\lambda} \end{aligned}$$

Q1. The time in minutes, X , between the arrival of successive customers at a post office is exponentially distributed with pdf $f(x) = 0.2e^{-0.2x}$.

a) What is the expected time between arrivals?

b) A customer arrives into the post office at 12.30 p.m. What is the probability the next customer arrives:
(i)on or before 12.32 p.m.? (ii)after 12.35 p.m

Solution:

a) The expected time between arrivals is
Given $\lambda = 0.2$

$$\mu = \frac{1}{0.2} = 5 \ minutes$$

b)

i) If the next customer arrives on or before 12.32 p.m., it means that the time between their arrival and the previous arrival is at most 2 minutes. Then

$$\begin{aligned} P(X \le 2) &= \int_0^2 \lambda e^{-\lambda x} dx = \int_0^2 0.2e^{-0.2x} dx \\ &= \left[\frac{0.2e^{-0.2x}}{-0.2}\right]_0^2 = \left[-e^{-0.2x}\right]_0^2 \\ &= 1 - e^{-0.2\times2} = 1 - 0.67032 = 0.33 \end{aligned}$$

ii) If the next customer arrives after 12.35 PM then the time between the two customers is more than 5 minutes.

$$\begin{aligned} P(X > 5) &= 1 - P(X \le 5) = 1 - (1 - e^{-0.2\times5}) \\ &= e^{-1} = 0.368 \end{aligned}$$

Q2. On the average, a certain computer part lasts ten years. The length of time the computer part lasts is exponentially distributed.

a) What is the probability that a computer part lasts more than 7 years?

b) On the average, how long would five computer parts last if they are used one after another?

c) Eighty percent of computer parts last at most how long?

d) What is the probability that a computer part lasts between 9 and 11 years?

Solution:

a) Let x=the amount of time (in years) a computer part lasts. $\mu = 10$ then λ

$$\lambda = \frac{1}{10} = 0.1$$

$$\begin{aligned} P(X > 7) &= 1 - P(X < 7) = 1 - (1 - e^{-0.1\times7}) \\ &= e^{-0.7} = 0.4966 \end{aligned}$$

The probability that a computer part lasts more than seven years is 0.4966
b) On the average, one computer part lasts ten years. Therefore, five computer parts, if they are used one after the other would last, on the average is

$$5 \times 10 = 50 years.$$

c. Let k = the 80 percentile.

$$\begin{aligned} P(X < k) &= 0.8 \\ 0.8 &= (1 - e^{-0.1\times k}) \\ e^{-0.1\times k} &= 1 - 0.8 = 0.2 \\ -0.1 \times k &= ln(0.2) \\ k &= \frac{ln(0.2)}{-0.1} \\ &= 16.1 \ years \end{aligned}$$

d). The probability that a computer part lasts between 9 and 11 years is

$$\begin{aligned} P(9 < X < 11) &= P(x < 11) - P(x < 9) \\ &= (1 - e^{-0.1\times11}) - (1 - e^{-0.1\times9}) \\ &= 0.6671 - 0.5934 = 0.0737 \end{aligned}$$

The probability that a computer part lasts between nine and 11 years is 0.0737.

Q3. On average, a pair of running shoes can last 18 months if used every day. The length of time running shoes last is exponentially distributed. What is the probability that a pair of running shoes last more than 15 months? On average, how long would six pairs of running shoes last if they are used one after the other? Eighty percent of running shoes last at most how long if used every day?

Solution:

Let x=the amount of time (in months) running shoes can last. $\mu = 18$ then λ

$$\lambda = \frac{1}{18} = 0.056$$

The probability that a pair of running shoes last more than 15 months is

$$\begin{aligned} P(X > 15) &= 1 - P(X < 15) = 1 - (1 - e^{-0.056\times 15}) \\ &= e^{-0.84} = 0.4317 \end{aligned}$$

Six pairs of running shoes would last for $18 \times 6 = 108$ months on average.
Let k = the 80 percentile.

$$\begin{aligned} P(X < k) &= 0.8 \\ 0.8 &= (1 - e^{0.056\times k}) \\ e^{-0.056\times k} &= 1 - 0.8 = 0.2 \\ -0.05 \times k &= ln(0.2) \\ k &= \frac{ln(0.2)}{0.0556} \\ &= 28.97\ months \end{aligned}$$

Q4. If jobs arrive every 15 seconds on average, $\lambda = 4$ per minute, what is the probability of waiting less than or equal to 30 seconds, i.e .5 min? What is the maximum waiting time between two job submissions with 95% confidence?

Solution:

$$\begin{aligned} P(\leq 0.5) &= \int_0^{0.5} \lambda e^{-\lambda t} dt = \int_0^{0.5} 4e^{-4t} dt \\ &= \left[\frac{4e^{-4t}}{-4}\right]_0^{0.5} \\ &= \left[-e^{-4t}\right]_0^{0.5} \\ &= 1 - e^{-4\times 0.5} = 1 - e^{-2} \\ &= 0.86 \end{aligned}$$

Let k = the 95 percentile.

$$\begin{aligned} P(X < k) &= 0.95 \\ 0.95 &= (1 - e^{-4\times k}) \\ e^{-4\times k} &= 1 - 0.95 = 0.05 \\ -4 \times k &= ln(0.05) \\ k &= \frac{ln(0.05)}{-4} \\ &= 0.748933 \end{aligned}$$

Q5. The time intervals between successive metro trains passing a certain station on a busy line have an exponential distribution with mean 8 minutes.

a) Find the probability that the time interval between two successive trains is less than 5 minutes.

b) Find a time interval t such that we can be 95% sure that the time interval between two successive trains will be greater than t

Solution:

$\mu = 8$ then $\lambda = 0.125$

$$\begin{aligned} P(T < 5) &= \int_0^5 \lambda e^{-\lambda t} dt = \int_0^5 0.125 e^{-0.125t} dt \\ &= \left[\frac{0.125e^{-0.125t}}{-0.125}\right]_0^5 \\ &= \left[-e^{-4t}\right]_0^5 \\ &= 1 - e^{-0.125\times 5} = 1 - e^{-0.625} \\ &= 0.4647. \end{aligned}$$

Let k = the 95 percentile.

$$\begin{aligned} P(X < k) &= 0.95 \\ 0.95 &= \int_t^{\infty} \lambda e^{-\lambda t} \\ &= e^{-\lambda t} = e^{-0.125t} \\ -0.125t &= ln(0.95) \\ t &= \frac{ln(0.95)}{-0.125} \\ &= 0.4103 \end{aligned}$$

Q6. Suppose that on a certain stretch of highway, cars pass at an average rate of five cars per minute. Assume that the duration of time between successive cars follows the exponential distribution.

a) On average, how many seconds elapse between two successive cars?

b) After a car passes by, how long on average will it take for another seven cars to pass by?

c) Find the probability that after a car passes by, the next car will pass within the next 20 seconds.

d) Find the probability that after a car passes by, the next car will not pass for at least another 15 seconds.

Solution:

On average, elapsed time in seconds between two successive cars is

a) Cars pass at an average rate of five cars per minute.

$$= \frac{60}{5} = 12 \textit{ seconds between two successive cars}$$

b) After a car passes by, on average the time taken for another seven cars to pass is

$$= 12 \times 7 = 84 \textit{ seconds next seven cars to pass}$$

c) The probability that after a car passes by, the next car will pass within the next 20 seconds is $\lambda = \frac{1}{12} = 0.0833$

$$\begin{aligned} P(T < 20) &= \int_0^{20} \lambda e^{-\lambda t} \\ &= \left[\frac{\lambda e^{-\lambda t}}{-\lambda}\right]_0^t \\ &= \left[1 - e^{-0.0833 \times 20}\right] \\ &= 1 - e^{-1.666} \\ &= 0.8111 \end{aligned}$$

d) The probability that after a car passes by, the next car will pass at least another 15 seconds is $\lambda = \frac{1}{12} = 0.0833$

$$\begin{aligned} P(T > 15) &= 1 - P(T < 15) \\ &= 1 - \int_0^{15} \lambda e^{-\lambda t} \\ &= 1 - \left[\frac{\lambda e^{-\lambda t}}{-\lambda}\right]_0^t \\ &= 1 - \left[1 - e^{-0.0833 \times 15}\right] \\ &= e^{-1.24995} \\ &= 0.2865 \end{aligned}$$

Q 7. The lifetime T (years) of an electronic component is a continuous random variable with a probability density function given by $f(t) = e^{-t} \quad t \geq 0$. Find the lifetime L which a typical component is 60% certain to exceed. If five components are sold to a manufacturer, find the probability that at least one of them will have a lifetime less than L years. [?]

Solution:

$$\lambda = 1$$

$$\begin{aligned} P(T > L) &= 0.6 \\ P(T > L) &= \int_L^{\infty} \lambda e^{-\lambda t} \\ &= \left[-e^{-t}\right]_L^{\infty} \\ &= e^{-L} \\ L &= -ln(0.6) = 0.51 \textit{ years} \\ &= 0.2865 \end{aligned}$$

P(at least one component has a lifetime less than 0.51 years)
= 1 - P(no component has a lifetime less than 0.51 years)

$$\begin{aligned} &= 1 - (0.6)^5 \\ &= 0.92 \end{aligned}$$

Q 8. It is assumed that the average time customers spends on hold when contacting a gas company's call centre is five minutes. What is the probability the a customer waits for longer than 15 minutes. If the the average waiting time is reduced to four minutes. What is the probability the a customer waits for longer than 15 minutes.

Solution: $\lambda = \frac{1}{5} = 0.2$

$$\begin{aligned} P(T > 15) &= 1 - P(T \leq 15) \\ &= 1 - \int_0^{15} \lambda e^{-\lambda t} \\ &= 1 - \left[\frac{\lambda e^{-\lambda t}}{-\lambda}\right]_0^t \\ &= 1 - \left[-e^{-\lambda t}\right]_0^t \\ &= 1 - \left[1 - e^{-0.2 \times 15}\right] \\ &= e^{-3} \\ &= 0.05 \end{aligned}$$

$\lambda = \frac{1}{4} = 0.25$

$$\begin{aligned} P(T > 15) &= 1 - P(T \leq 15) \\ &= 1 - \int_0^{15} \lambda e^{-\lambda t} \\ &= 1 - \left[\frac{\lambda e^{-\lambda t}}{-\lambda}\right]_0^t \\ &= 1 - \left[-e^{-\lambda t}\right]_0^t \\ &= 1 - \left[1 - e^{-0.25 \times 15}\right] \\ &= e^{-3.75} \\ &= 0.024 \end{aligned}$$

Note: Entire material is taken from different text books or from the Internet (different websites). Slightly it is modified from the original content. It is not for any commercial purpose. It is used to teach students. Suggestions are always encouraged.

1.8 The Normal Gaussian Distribution: [?, ?, ?]

The normal distribution is one of the most familiar probability distributions in statistics. It is also called as Gaussian distribution. It is a continuous distribution, this means that it takes values in the range between $-\infty$ to ∞. The probability density function(pdf) of normal distribution is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty \leq X \leq -\infty$$

where

$f(x)$ = frequency of random variable X
μ = mean
σ = standard deviation
x = value of random variable

- It is a bell shaped curve and is symmetrical about its mean μ.
- σ is called as shape parameter i.e., its shape is varied as σ varies.
- The distribution function $f(x)$ decreases rapidly as x increases.
- The maximum probability $f(x)$ occurs at $x = \mu$ and its value is $f(x) = \frac{1}{\sigma^2\sqrt{2\pi}}$.
- The mean, median and mode are coincide with each other.
- The area under the curve and over the x axis is unity.
- The random variable x can take values from $-\infty$ to ∞
- The area under the curve for $x = \mu \pm \sigma$ is as follows:

  1. The area under the curve for $x = \mu - \sigma$ and $x = \mu + \sigma$ is 0.6826
  2. The area under the curve for $x = \mu - 2\sigma$ and $x = \mu + 2\sigma$ is 0.9544
  3. The area under the curve for $x = \mu - 3\sigma$ and $x = \mu + 3\sigma$ is 0.9973

The details area under the curve for $x = \mu \pm \sigma$ is as shown in Figure 1.17

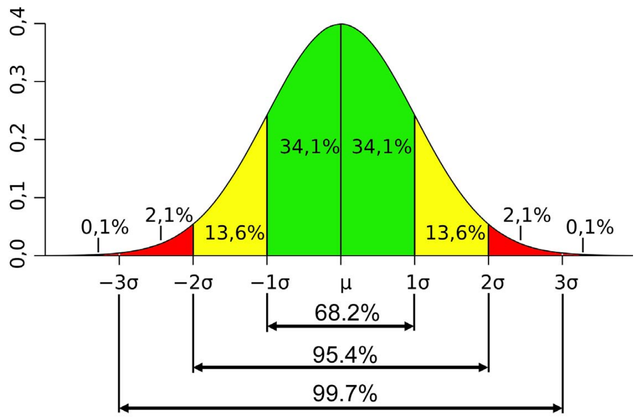

Figure 1.17: The normal curve.

The details of the mean and variance is as shown in Figure 1.18, Figure1.19, Figure1.20

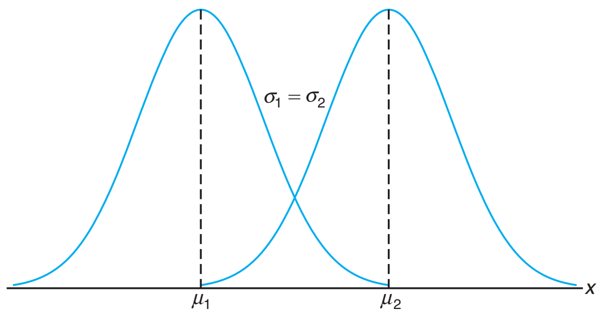

Figure 1.18: Normal curves with $\mu_1 < \mu_2$ and $\sigma_1 = \sigma_2$.

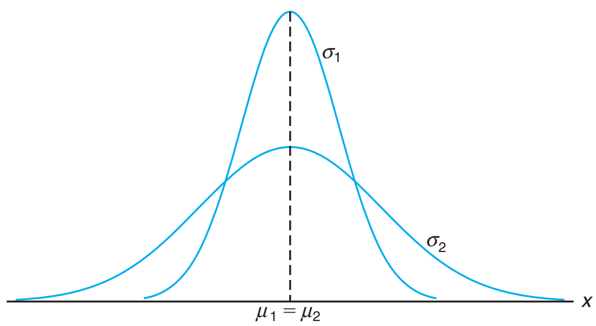

Figure 1.19: Normal curves with $\mu_1 = \mu_2$ and $\sigma_1 < \sigma_2$.

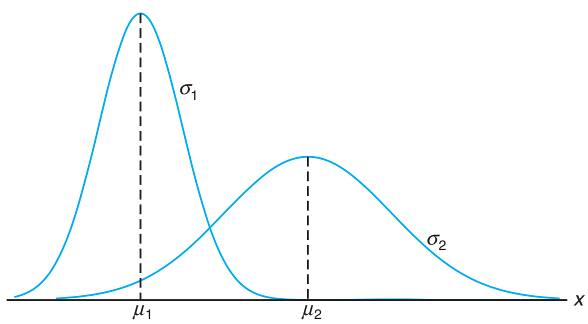

Figure 1.20: Normal curves with $\mu_1 < \mu_2$ and $\sigma_1 < \sigma_2$.

Applications of the Normal Distribution

- Height of people
- Scientific measurements
- Students average marks scored.
- Amount of rain fall.
- Birth Weight

Cumulative Distribution Function (cdf):

$$F(x) = \int f(x)dx = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$\begin{aligned} u &= \frac{x-\mu}{\sigma} \\ x-\mu &= \sigma u \\ dx &= \sigma du \end{aligned}$$

$$\begin{aligned} F(x) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{u} e^{-\frac{1}{2}(u)^2} \sigma du \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-\frac{1}{2}(u)^2} du \\ &= \phi\left(\frac{x-\mu}{\sigma}\right) \end{aligned}$$

$\phi\left(\frac{x-\mu}{\sigma}\right)$ is evaluated by the normal standard table.

Mean:

$$\begin{aligned} E[X] = \mu &= \int_{-\infty}^{\infty} x f(x)dx \\ &= \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \end{aligned}$$

$$\begin{aligned} z &= \frac{x-\mu}{\sigma} \\ z\sigma &= x-\mu \\ dx = \sigma dz \\ x &= \mu + \sigma z \end{aligned}$$

Limits when $x = -\infty \quad z = -\infty$ and $x = \infty \quad z = \infty$

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z) e^{-\frac{1}{2}z^2} \sigma dz \\ &= \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz + \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-\frac{1}{2}z^2} dz \\ &= \frac{\mu}{\sqrt{2\pi}} 2 \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz + 0 \end{aligned}$$

$$\int_{-\infty}^{\infty} z e^{-\frac{1}{2}z^2} dz = 0 \quad \because It\ is\ an\ odd$$

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz = \sqrt{\frac{\pi}{2}} \quad \because It\ is\ an\ even$$

$$\begin{aligned} E[X] &= \frac{2\mu}{\sqrt{2\pi}} \sqrt{\frac{\pi}{2}} \\ &= \mu \end{aligned}$$

Variance:

$$Var[X] = E[X^2] - [E(X)]^2 = E(x-\mu)^2$$

$$\begin{aligned} E(x-\mu)^2 &= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx \\ &= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \end{aligned}$$

$$\begin{aligned} z &= \frac{x-\mu}{\sigma} \\ z\sigma &= x-\mu \\ dx &= \sigma dz \end{aligned}$$

Limits when $x = -\infty \quad z = -\infty$ and $x = \infty \quad z = \infty$

$$\begin{aligned} Var[X] &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma^2 z^2 e^{-\frac{1}{2}z^2} \sigma dz \\ &= \frac{\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} z^2 e^{-\frac{1}{2}z^2} dz \end{aligned}$$

$$\begin{aligned} \frac{z^2}{2} &= t \\ zdz &= dt \\ z = \sqrt{2t} \end{aligned}$$

$$\begin{aligned} Var[X] &= \frac{\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} z^2 e^{-\frac{1}{2}z^2} dz \\ &= \frac{2\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} 2t e^{-t^2} \frac{dt}{\sqrt{2t}} \\ &= \frac{2\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} t^{\frac{1}{2}} e^{-t^2} dt \\ &= \frac{2\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} t^{\frac{3}{2}-1} e^{-t^2} dt \end{aligned}$$

$$\int_{0}^{\infty} t^{\frac{3}{2}-1} e^{-t^2} dt = \Gamma\left(\frac{3}{2}\right)$$

$$\begin{aligned} Var[X] &= \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) \\ &= \frac{2\sigma^2}{\sqrt{\pi}} \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \\ &= \frac{2\sigma^2}{\sqrt{\pi}} \frac{1}{2} \sqrt{\pi} \\ &= \sigma^2 \end{aligned}$$

Q 6.4. Given a random variable X having a normal distribution with $\mu = 50$ and $\sigma = 10$, find the probability that X assumes a value between 45 and 62.

Solution:

$$z = \frac{x - \mu}{\sigma}$$

The z values corresponding to $x_1 = 45$ and $x_2 = 62$ are

$$z_1 = \frac{45 - 50}{10} = -0.5 \qquad z_2 = \frac{62 - 50}{10} = 1.2$$

$$\begin{aligned} P(45 < x < 62) &= P(-0.5 < x < 1.2) = P(z < 1.2) - P(z < -0.5) \\ &= 0.8849 - 0.3085 = 0.5764 \end{aligned}$$

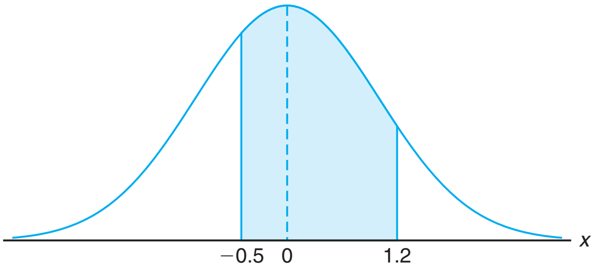

Figure 1.21

The values of Z(1.2) and Z(-0.5) are determined from the Z table. The details are as shown in Figure 1.22 &1.23

Table A.3 (continued) Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115

Figure 1.22

−1.4	0.0808	0.0793	0.0778	0.0764
−1.3	0.0968	0.0951	0.0934	0.0918
−1.2	0.1151	0.1131	0.1112	0.1093
−1.1	0.1357	0.1335	0.1314	0.1292
−1.0	0.1587	0.1562	0.1539	0.1515
−0.9	0.1841	0.1814	0.1788	0.1762
−0.8	0.2119	0.2090	0.2061	0.2033
−0.7	0.2420	0.2389	0.2358	0.2327
−0.6	0.2743	0.2709	0.2676	0.2643
−0.5	0.3085	0.3050	0.3015	0.2981
−0.4	0.3446	0.3409	0.3372	0.3336
−0.3	0.3821	0.3783	0.3745	0.3707

Figure 1.23

Q 6.5. Given a random variable X having a normal distribution with $\mu = 300$ and $\sigma = 50$, find the probability that X assumes a value greater than 362.

Solution:

The z values corresponding to $x = 362$ is

$$z = \frac{362 - 300}{50} = 1.24$$

$$\begin{aligned} P(X > 362) &= P(Z > 1.24) = 1 - P(Z < 1.24) \\ &= 1 - 0.8975 = 0.1075 \end{aligned}$$

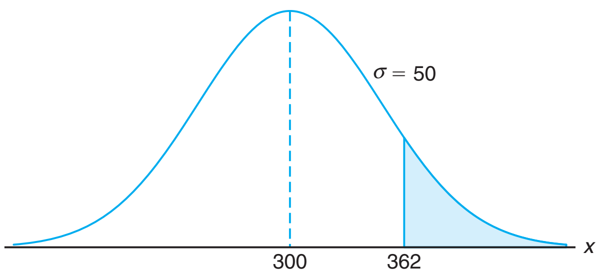

Figure 1.24

Q 6.6. Given a normal distribution with $\mu = 40$ and $\sigma = 6$ find the value of x that has
(a) 45% of the area to the left and
(b) 14% of the area to the right.

Solution:

An area of 0.45 to the left of the desired x value is shaded as shown in Figure. From the table the value of z from the table is 0.45

$$z = \frac{x-\mu}{\sigma}$$

$$\begin{aligned} x &= z\sigma + \mu = 6 \times (-0.13) + 40 \\ &= 39.22 \end{aligned}$$

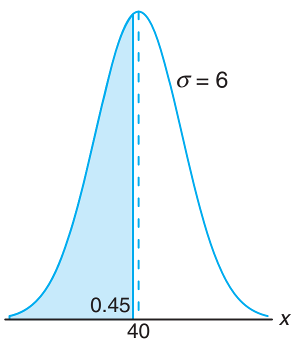

Figure 1.25

b) An area equal to 0.14 to the right of the desired x value. The z value that leaves 0.14 of the area to the right and hence an area of 0.86 (1-0.14=0.86) to the left. From Table A.3, $P(Z < 1.08) = 0.86$, so the desired z value is 1.08.

$$z = \frac{x-\mu}{\sigma}$$

$$\begin{aligned} x &= z\sigma + \mu = 6 \times (1.08) + 40 \\ &= 46.48 \end{aligned}$$

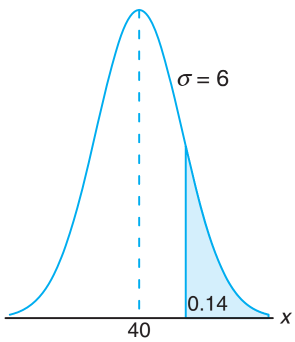

Figure 1.26

Q 6.7. A certain type of storage battery lasts, on average, 3.0 years with a standard deviation of 0.5 year. Assuming that battery life is normally distributed, find the probability that a given battery will last less than 2.3 years.

Solution:

The probability that a given battery will last less than 2.3 years. Hence $P(X < 2.3)$, evaluate the area under the normal curve to the left of 2.3. This can be done by finding the area to the left of the corresponding z value.

$$z = \frac{2.3 - 3}{0.5} = -1.4$$

$$\begin{aligned} P(X < 2.3) &= P(Z < 1.4) \\ &= 0.0808 \end{aligned}$$

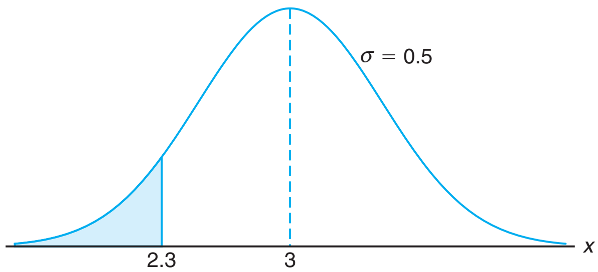

Figure 1.27

Q 6.8. An electrical firm manufactures light bulbs that have a life, before burn-out, that is normally distributed with mean equal to 800 hours and a standard deviation of 40 hours. Find the probability that a bulb burns between 778 and 834 hours.

Solution:

The z values corresponding to $x_1 = 778$ and $x_2 = 834$ are

$$z_1 = \frac{778 - 800}{40} = -0.55 \qquad z_2 = \frac{834 - 800}{40} = 0.85$$

$$\begin{aligned} P(778 < x < 834) &= P(-0.55 < x < 0.85) = P(z < 0.85) - P(z < -0.55) \\ &= 0.8023 - 0.2912 = 0.5111 \end{aligned}$$

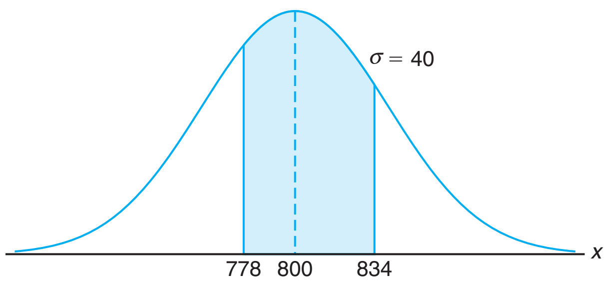

Figure 1.28

Determining the values of Z for more than two decimals

Example 1: Determining the values of Z $P(Z < 1.456)$ The Z table is supports upto two decimals. In this case it is upto 1.4. The value of the 1.456 is determined by interpolation method. The procedure is as follows.

P(1.456) is between 1.45 and 1.46, the difference is 0.01. The difference between 1.456 and 1.45 is 0.006.

$$\frac{0.006}{0.01} = 0.6$$

$$\begin{aligned} P(1.456) &= P(1.45) + 0.6[P(1.46) - P(1.45)] \\ &= 0.9265 + 0.6[0.9279 - 0.9265] = 0.9265 + 0.0084 \\ &= 0.9273 \end{aligned}$$

Example 2: Determining the values of Z $P(Z > 1.165)$

P(1.165) is between 1.16 and 1.17, the difference is 0.01. The difference between 1.16 and 1.165 is 0.005.

$$\frac{0.005}{0.01} = 0.5$$

$$\begin{aligned} P(Z > 1.165) &= P(1.16) + 0.5[P(1.17) - P(1.16)] \\ &= 0.8770 + 0.5[0.8790 - 0.8770] = 0.8770 + 0.0001 \\ &= 0.8771 \end{aligned}$$

Q 5. The random variable Z is normalized Gaussian (i.e., its mean is 0 and its variance is 1). Find the probability $P\{-0.505 < Z \le 1.456\}$ [?]

Solution:

$$\begin{aligned} P(-0.505 < Z \le 1.456) &= P(z < 1.456) - P(z < -0.505) \\ &= 0.9273 - 0.3067 = 0.6206 \end{aligned}$$

P(1.456) is between 1.45 and 1.46, the difference is 0.01. The difference between 1.45 and 1.456 is 0.006.

$$\frac{0.006}{0.01} = 0.6$$

$$\begin{aligned} P(1.456) &= P(1.45) + 0.6[P(1.46) - P(1.45)] \\ &= 0.9265 + 0.6[0.9279 - 0.9265] = 0.9265 + 0.0008 \\ &= 0.9273 \end{aligned}$$

P(-0.505) is between 0.50 and 0.51, the difference is 0.01. The difference between 0.50 and 0.0.505 is 0.005.

$$\frac{0.005}{0.01} = 0.5$$

$$\begin{aligned} P(-0.505) &= P(-0.50) + 0.5[P(-0.51) - P(-0.50)] \\ &= 0.3085 + 0.5[0.3050 - 0.3085] = 0.3085 - 0.0018 \\ &= 0.3067 \end{aligned}$$

Q 6. The random variable Z is normalized Gaussian (i.e., its mean is 0 and its variance is 1). Find the probability $P\{-1.003 < Z \le 2.392\}$ [?]

Solution:

$$\begin{aligned} P(-1.003 < Z \le 2.392) &= P(z < 2.392) - P(z < -1.003) \\ &= 0.9916 - 0.1579 = 0.8337 \end{aligned}$$

P(2.392) is between 2.39 and 2.40, the difference is 0.01. The difference between 2.39 and 2.392 is 0.002.

$$\frac{0.002}{0.01} = 0.2$$

$$\begin{aligned} P(2.392) &= P(2.39) + 0.2[P(2.40) - P(2.39)] \\ &= 0.9916 + 0.2[0.9918 - 0.9916] = 0.9916 + 0.0000 \\ &= 0.9916 \end{aligned}$$

P(-1.003) is between -1.00 and -1.01, the difference is 0.01. The difference between -1.00 and -1.003 is 0.003.

$$\frac{0.003}{0.01} = 0.3$$

$$\begin{aligned} P(-1.003) &= P(-1.00) + 0.3[P(-1.01) - P(-1.00)] \\ &= 0.1587 + 0.3[0.1562 - 0.1587] = 0.1587 - 0.0008 \\ &= 0.1579 \end{aligned}$$

Q 7. The random variable Z is normalized Gaussian (i.e., its mean is 0 and its variance is 1). Find the probability $P\{-2.335 < Z \leq 0.595\}$ [?]

Solution:

P(0.595) is between 0.59 and 0.60, the difference is 0.01. The difference between 0.59 and 0.595 is 0.005.

$$\frac{0.005}{0.01} = 0.5$$

$$\begin{aligned} P(0.595) &= P(0.59) + 0.5[P(0.60) - P(0.59)] \\ &= 0.7224 + 0.5[0.7257 - 0.7224] = 0.7224 + 0.0017 \\ &= 0.7241 \end{aligned}$$

P(-2.335) is between -2.33 and -2.34, the difference is 0.01. The difference between -2.33 and -2.335 is 0.005.

$$\frac{0.005}{0.01} = 0.5$$

$$\begin{aligned} P(-2.335) &= P(-2.33) + 0.5[P(-2.34) - P(-2.33)] \\ &= 0.0099 - 0.5[0.0096 - 0.0099] = 0.0099 - 0.0001 \\ &= 0.0098 \end{aligned}$$

$$\begin{aligned} P(-2.335 < Z \leq 0.595) &= P(z < 0.595) - P(z < -2.335) \\ &= 0.7241 - 0.0098 = 0.7143 \end{aligned}$$

Q 61. It is given that the random variable Z is Gaussian with mean of 0 and a variance of 1. The random variable Y is obtained from Z with the relation $y = 3z + 7$. Find the pdf for Y. [?]

Solution:

$$\begin{aligned} y &= 3z + 7 \\ z &= \frac{y-7}{3} \end{aligned}$$

$$\begin{aligned} F(z) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \\ F(Y) &= P\{Y \leq y\} = P\left[Z \leq \frac{y-7}{3}\right] = F\left[\frac{y-7}{3}\right] \\ f(Y) &= \frac{d}{dy} F\left[\frac{y-7}{3}\right] = f\left[\frac{y-7}{3}\right]\frac{1}{3} \\ &= \frac{1}{3\sqrt{2\pi}} e^{-\frac{(y-7)^2}{18}} \end{aligned}$$

Q 62. It is given that the random variable X is Gaussian with mean of 0 and a variance of 1. The random variable Y is obtained from X with the relation $y = 5x - 6$. Find the pdf for Y. [?]

Solution:

$$\begin{aligned} y &= 5x - 6 \\ x &= \frac{y+6}{5} \end{aligned}$$

$$\begin{aligned} F(z) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \\ F(Y) &= P\{Y \leq y\} = P\left[Z \leq \frac{y+6}{5}\right] = F\left[\frac{y+6}{5}\right] \\ f(Y) &= \frac{d}{dy} F\left[\frac{y+6}{5}\right] = f\left[\frac{y+6}{5}\right]\frac{1}{5} \\ &= \frac{1}{5\sqrt{2\pi}} e^{-\frac{(y+6)^2}{50}} \end{aligned}$$

Q 64. X is a Gaussian random variable with a mean of 75.1 and standard deviation of 10.2. Evaluate $P\{59.9 < X < 81.75\}$ [?]

Solution:

$$\begin{aligned} P\{59.9 < X < 81.75\} &= F_X(81.75) - F_X(59.9) \\ z &= \frac{x-\mu}{\sigma} \\ &= Z\left[\frac{81.75 - 75.1}{10.2}\right] - Z\left[\frac{59.9 - 75.1}{10.2}\right] \\ &= Z(0.6520) - Z(-1.4902) \\ &= 0.7428 - 0.0681 \\ &= 0.6747 \end{aligned}$$

P(0.6520) is between 0.65 and 0.66, the difference is 0.01. The difference between 0.65 and 0.6520 is 0.0020.

$$\frac{0.0020}{0.01} = 0.2$$

$$\begin{aligned} P(0.6520) &= P(0.65) + 0.2[P(66) - P(0.65)] \\ &= 0.7422 + 0.2[0.7454 - 0.7422] = 0.7422 + 0.0006 \\ &= 0.7428 \end{aligned}$$

Q 65. X is a Gaussian random variable with a mean of 73.16 and standard deviation of 20.35. What is the probability that X occurs between 50 and 80? [?]

Solution:

$$\begin{aligned} P\{50 < X < 80\} &= F_X(80) - F_X(50) \\ z &= \frac{x-\mu}{\sigma} \\ &= Z\left[\frac{80 - 73.16}{20.35}\right] - Z\left[\frac{50 - 73.16}{20.35}\right] \\ &= Z(0.3361) - Z(-1.1381) \\ &= 0.6316 - 0.1275 \\ &= 0.5041 \end{aligned}$$

P(0.3361) is between 0.33 and 0.34, the difference is 0.01. The difference between 0.33 and 0.3361 is 0.0061.

$$\frac{0.0061}{0.01} = 0.61$$

$$\begin{aligned} P(0.3361) &= P(0.33) + 0.61[P(0.34) - P(0.33)] \\ &= 0.6293 + 0.61[0.6331 - 0.6293] = 0.6293 + 0.0023 \\ &= 0.6316 \end{aligned}$$

P(-1.1381) is between -1.13 and -1.14, the difference is 0.01. The difference between -1.13 and -1.1381 is 0.0081.

$$\frac{0.0081}{0.01} = 0.81$$

$$\begin{aligned} P(-1.1381) &= P(-1.13) + 0.61[P(0.34) - P(-1.13)] \\ &= 0.1292 - 0.81[0.1271 - 0.1292] = 0.1292 - 0.0017 \\ &= 0.1275 \end{aligned}$$

Q 66. X is a Gaussian random variable with a mean of $\mu_X = 73.16$ and $\sigma_X = 20.35$. Evaluate $P\{53.5 < X < 80.9\}$ [?]

Solution:

$$\begin{aligned} P\{53.5 < X < 80.9\} &= F_X(80.9) - F_X(53.5) \\ z &= \frac{x - \mu}{\sigma} \\ &= Z\left[\frac{80.9 - 73.16}{20.35}\right] - Z\left[\frac{53.5 - 73.16}{20.35}\right] \\ &= Z(0.3803) - Z(-0.9661) \\ &= 0.6481 - 0.16970 \\ &= 0.4812 \end{aligned}$$

Q 67. A dimension and its tolerance are specified to be $y = 0.75 \pm 0.002$ in. Y is a Gaussian random variable with a mean of 0.751 in and standard deviation of 0.0013 in. Estimate the percent of realization of Y that will be within the tolerance range. [?]

Solution:

$$\begin{aligned} P\{0.75 \pm 0.002\} &= P\{0.75 - 0.002 < Y < 0.75 + 0.002\} \\ &= F_Y(0.752) - F_Y(0.748) \\ z &= \frac{x - \mu}{\sigma} \\ &= Z\left[\frac{0.752 - 0.751}{0.0013}\right] - Z\left[\frac{0.748 - 0.751}{0.0013}\right] \\ &= Z(0.7692) - Z(-2.3077) \\ &= 0.7791 - 0.0105 = 0.7686 \\ &= 76.86\% \end{aligned}$$

Q 68. A production drawing specifies a dimension on printed circuit board and as 0.75 ± 0.002 cm. Assume that the process that manufactures the devices is such that that dimension is a Gaussian random variable Y and that the manufacturing process is slightly inaccurate $\mu_Y = 0.7505$. Assume that the precision of the manufacturing process is such that the standard deviation is $\sigma_Y = 0.0014$ cm. What percent of printed circuit boards have the dimension within the specified tolerance limits? [?]

Solution:

$$
\begin{aligned}
P\{0.75 \pm 0.002\} &= P\{0.75 - 0.002 < Y < 0.75 + 0.002\} \\
&= F_Y(0.752) - F_Y(0.748) \\
z &= \frac{x-\mu}{\sigma} \\
&= Z\left[\frac{0.752 - 0.7505}{0.0014}\right] - Z\left[\frac{0.748 - 0.7505}{0.0014}\right] \\
&= Z(1.0714) - Z(-1.7857) \\
&= 0.8580 - 0.0371 = 0.8209 \\
&= 82.09\%
\end{aligned}
$$

Q 69. A dimension and its tolerance are specified to be $y = 0.75 \pm 0.002$ in. Y is a Gaussian random variable with a man of $\mu_Y = 0.7495$ in and standard deviation of $\sigma_Y = 0.0013$ in. Estimate the percent of realizations of Y that will be within the tolerance range. [?]

Solution:

$$
\begin{aligned}
P\{0.75 \pm 0.002\} &= P\{0.75 - 0.002 < Y < 0.75 + 0.002\} \\
&= F_Y(0.752) - F_Y(0.7495) \\
z &= \frac{x-\mu}{\sigma} \\
&= Z\left[\frac{0.752 - 0.7495}{0.0013}\right] - Z\left[\frac{0.748 - 0.7495}{0.0013}\right] \\
&= Z(1.9231) - Z(-1.538) \\
&= 0.9728 - 0.1243 = 0.8485 \\
&= 84.85\%
\end{aligned}
$$

Q 74. The random variable X is Gaussian $\mu_X = 0$ and $\sigma_X = 1$. The random variable Y is obtained from X using $y = 10x^3$. What is $F_Y(0.1298)$ [?]

Solution:

$$
\begin{aligned}
y &= 10x^3 \\
x &= \left[\frac{y}{10}\right]^{\frac{1}{3}} \\
F_Y(0.1298) &= P\{< 0.2530\} \\
Z(0.2530) &= 0.5929
\end{aligned}
$$

Q 2. The weights of 300 students are normally distributed with mean 68 Kgs and standard deviation 3 kgs. How many students have masses (i)greater than 72 kgs (ii) less than or equal to 64 kgs. (iii) between 65 and 71 kgs inclusive [?]

Solution:
(i) $P(X > 72)$ The number of students having masses greater than 72

$$
\begin{aligned}
z &= \frac{x-\mu}{\sigma} \\
&= Z\left[\frac{72-68}{3}\right] = Z(1.33) \\
P(X > 72) &= P(Z > 1.33) = 1 - P(Z \le 1.33) \\
&= 1 - F(1.33) = 1 - 0.9082 = 0.0918
\end{aligned}
$$

The number of students having masses greater than 72 are

$$\begin{aligned} &= N \times P(X > 72) \\ &= 300 \times 0.0918 = 27.54 \\ &\approx 28 \end{aligned}$$

(ii) $P(X \leq 64)$ The number of students having masses less than or equal to 72

$$\begin{aligned} z &= \frac{x-\mu}{\sigma} \\ &= Z\left[\frac{64-68}{3}\right] = Z(-1.33) \\ P(X \leq 64) &= P(Z \leq -1.33) \\ &= F(-1.33) = 0.9082 \end{aligned}$$

The number of students having masses less than or equal to 72 are

$$\begin{aligned} &= N \times P(X \leq 64) \\ &= 300 \times 0.9082 = 27.246 \\ &\approx 27 \end{aligned}$$

(iii) $P(65 \leq X \leq 71)$ The number of students between 65 and 71 kgs inclusive

$$\begin{aligned} z &= \frac{x-\mu}{\sigma} \\ P(65 \leq X \leq 71) &= Z\left[\frac{71-68}{3}\right] - Z\left[\frac{65-68}{3}\right] = Z(-1) - Z(1) \\ &= 0.8413 - 0.1587 \\ &= 0.6826 \end{aligned}$$

The number of students between 65 and 71 kgs inclusive are

$$\begin{aligned} &= N \times P(X \leq 64) \\ &= 300 \times 0.6826 = 204.78 \\ &\approx 205 \end{aligned}$$

Q 3. The heights of 100 students in a class is 158 cms and normally distributed with standard deviation of 20 cms. Find how many students heights lie between 150 cms and 170 cms. [?]

Solution:
$P(150 < X < 170)$ The number of students between 150 and 170 cms

$$\begin{aligned} z &= \frac{x-\mu}{\sigma} \\ P(150 < X < 170) &= Z\left[\frac{150-158}{20}\right] - Z\left[\frac{170-158}{20}\right] \\ P(-0.4 < X < 0.6) &= Z(0.6) - Z(-0.4) = 0.7257 - 0.3446 \\ &= 0.3811 \end{aligned}$$

The number of students between 150 and 170 cms are

$$\begin{aligned} &= N \times P(150 < X < 170) \\ &= 100 \times 0.3811 = 38.11 \\ &\approx 38 \end{aligned}$$

Q 5. In a sample of 1000 cases, the mean of a certain test is 14 and standard evasion is 2.5. Assuming the distribution is normal Find (i) how many students score between 12 and 15 (ii) how many score above 18 (iii) how many score below 18. [?]

Solution:
$P(12 < X < 15)$ The score between 12 and 15

$$\begin{aligned} z &= \frac{x-\mu}{\sigma} \\ P((12 < X < 15) &= Z\left[\frac{12-14}{2.5}\right] - Z\left[\frac{15-14}{2.5}\right] \\ P(-0.8 < X < 0.4) &= Z(0.4) - Z(-0.8) = 0.6554 - 0.2119 \\ &= 0.4435 \end{aligned}$$

The score between 12 and 15 are

$$\begin{aligned} &= N \times P(12 < X < 15) \\ &= 1000 \times 0.4435 = 443.5 \\ &\approx 443 \; or \; 444 \end{aligned}$$

(ii) $P(X > 18)$ the score above 18

$$\begin{aligned} z &= \frac{x-\mu}{\sigma} \\ P(X > 18) &= Z\left[\frac{18-14}{2.5}\right] = 1.6 \\ P(X > 18) &= 1 - P(X \leq 18) = 1 - P(Z \leq 1.6) \\ &= 1 - Z(1.6)1 - 0.9452 \\ &= 0.0548 \end{aligned}$$

The number of students score score above 18 are

$$\begin{aligned} &= N \times P(X > 18) \\ &= 1000 \times 0.0548 = 54.8 \\ &\approx 55 \end{aligned}$$

(iii) $P(X < 18)$ the score below 18

$$\begin{aligned} z &= \frac{x-\mu}{\sigma} \\ P(X < 18) &= Z\left[\frac{18-14}{2.5}\right] = 1.6 \\ &= P(Z \leq 1.6) \\ &= Z(1.6) = 0.9452 \end{aligned}$$

The number of students score score above 18 are

$$\begin{aligned} &= N \times P(X < 18) \\ &= 1000 \times 0.9452 = 945.2 \\ &\approx 945 \end{aligned}$$

Q 24. In a distribution exactly normal 7% of the items are under 35 and 89% are under 63. What are the mean and standard deviation of the distribution. [?]

Solution:
$P(X \leq 35) = 0.07$ $P(X \leq 63) = 0.89$

When X=35

$$z = \frac{x-\mu}{\sigma} = \frac{35-\mu}{\sigma}$$

When X=63

$$z = \frac{63-\mu}{\sigma} = \frac{35-\mu}{\sigma}$$

$$\begin{aligned} P(X \le 35) &= 0.07 \\ P\left(Z \le \left[\frac{35-\mu}{\sigma}\right]\right) &= 0.07 \\ \frac{35-\mu}{\sigma} &= -1.48 \\ 35-\mu &= -1.48\sigma \end{aligned}$$

$$\mu - 1.48\sigma = 35 \tag{1.1}$$

$$\begin{aligned} P(X \le 63) &= 0.89 \\ P\left(Z \le \left[\frac{63-\mu}{\sigma}\right]\right) &= 0.89 \\ \frac{63-\mu}{\sigma} &= 1.23 \\ 63-\mu &= 1.23\sigma \end{aligned}$$

$$\mu + 1.23\sigma = 63 \tag{1.2}$$

Solving Equation 1.1 and 1.2

$$\mu = 50.3 \quad \sigma = 10.3$$

Q 26. The mean and standard deviation of normal variate are 8 and 4 respectively. Find (i) $P(5 \le X \le 10)$ **(i)** $P(X \ge 5)$ [?]

Solution:
(i) $P(5 \le X \le 10)$

$$\begin{aligned} z &= \frac{x-\mu}{\sigma} \\ z_1 &= \frac{5-8}{4} = -0.75 \\ z_2 &= \frac{10-8}{4} = 0.5 \end{aligned}$$

$$\begin{aligned} P(5 \le X \le 10) &= P(-0.75 < Z < 0.5) \\ &= Z(0.5) - Z(-0.75) = 0.6915 - 0.2266 \\ &= 0.4649 \end{aligned}$$

(ii) $P(X \ge 5)$

$$\begin{aligned} z &= \frac{x-\mu}{\sigma} = \frac{5-8}{4} \\ P(X \ge 5) &= 1 - P(Z \le -0.75) \\ &= 1 - Z(-0.75) = 1 - 0.2266 \\ &= 0.7734 \end{aligned}$$

1.9 Mean, Variance and standard deviation

1.9.1 Mean, Variance and Standard deviation for discrete random variable:

Discrete random variables:

The mean or expected value of a random variable X is:

$$\mu = E(X) = \sum_{i=1}^{R} x_i Pr(X = x_i) = \sum_{i=1}^{R} x_i f(x_i)$$

The variance of a random variable X is:

$$\begin{aligned}\sigma^2 = Var(X) &= \sum_{i=1}^{R}(x-\mu)^2 Pr(X = x_i) = \sum_{i=1}^{R}(x-\mu)^2 f(x_i) \\ &= E(X^2) - [E(X)]^2\end{aligned}$$

The standard deviation of a random variable X is:$\sigma = \sqrt{\sigma^2}$

$$V(X+a) = V(X) = E(X^2) - [E(x)]^2 \quad \textit{where a is constant.}$$

Proof

$$\begin{aligned}V(X+a) &= E[(X+a)^2] - [E(X+a)]^2 = E[X^2 + 2aX + a^2] - [E[X] + E[a]]^2 \\ &= E(X^2) + 2aE(X) + a^2 - \{[E(x)]^2 + 2aE(X) + a^2\} \\ &= E(X^2) + 2aE(X) + a^2 - [E(x)]^2 - 2aE(X) - a^2 \\ &= E(X^2) - [E(x)]^2\end{aligned}$$

1.9.2 Mean, Variance and Standard deviation of a continuous random variables:

Continuous random variables:

The mean or expected value of a random variable X is:

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)dx$$

The variance of a random variable X is:

$$\sigma^2 = Var(V) = \int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx = E(x^2) - [E(x)]^2$$

The standard deviation of a random variable X is:

$$\sigma = \sqrt{\sigma^2}$$

Proof

$$\begin{aligned}E(x-\mu)^2 &= E(x^2 - 2\mu x + \mu^2) \\ &= E(x^2) - 2E(x)\mu + \mu^2 = E(x^2) - 2\mu \times \mu + \mu^2 \\ &= E(x^2) - 2\mu^2 + \mu^2 = E(x^2) - \mu^2 \\ &= E(x^2) - [E(x)]^2\end{aligned}$$

Q 2 The random variable X has the following distribution which is as shown in table, Determine (a) k (b) mean (c) Variance [?]

X	-2	-1	0	1	2	3
P(X)	0.1	k	0.2	2k	0.3	k

Solution: (a)

$$\begin{aligned}\sum P(X) &= 1 \\ &= 0.1 + k + 0.2 + 2k + 0.3 + k \\ 1 &= 0.6 + 4k \\ k &= 0.1\end{aligned}$$

X	-2	-1	0	1	2	3
P(X)	0.1	0.1	0.2	0.2	0.3	0.1

(b) Mean

$$
\begin{aligned}
E[X] &= \sum_{i=1}^{n} x_i p(x_i) \\
&= -2 \times 0.1 + -1 \times 0.1 + 0 \times 0.2 + 1 \times 0.2 + 2 \times 0.3 + 3 \times 0.1 \\
&= 0.8
\end{aligned}
$$

(c) Variance

$$
\begin{aligned}
V[X] &= E(X^2) - [E(X)^2] \\
E(X^2) &= \sum_{i=1}^{n} x_i^2 p(x_i) \\
&= (-2)^2 \times 0.1 + (-1)^2 \times 0.1 + (0)^2 \times 0.2 + (1)^2 \times 0.2 + (2)^2 \times 0.3 + (3)^2 \times 0.1 \\
&= 0.4 + 0.1 + 0.2 + 1.2 + 0.9 = 2.8 \\
V[X] &= 2.8 - (0.8)^2 \\
&= 2.16
\end{aligned}
$$

Q 3 Find the variance of the discrete random variable X whose probability distribution is as shown in table.[?]

X	1	2	3	4	5
P(X)	0.1	0.1	0.3	0.3	0.2

Solution:

Mean

$$
\begin{aligned}
E[X] &= \sum_{i=1}^{n} x_i p(x_i) \\
&= 1 \times 0.1 + 2 \times 0.1 + 3 \times 0.3 + 4 \times 0.3 + 5 \times 0.2 \\
&= 3.4
\end{aligned}
$$

(c) Variance

$$
\begin{aligned}
V[X] &= E(X^2) - [E(X)^2] \\
E(X^2) &= \sum_{i=1}^{n} x_i^2 p(x_i) \\
&= (1)^2 \times 0.1 + (2)^2 \times 0.1 + (3)^2 \times 0.3 + (4)^2 \times 0.3 + (5)^2 \times 0.2 \\
&= 0.1 + 0.4 + 2.7 + 4.8 + 5 = 13 \\
V[X] &= 13 - (3.4)^2 \\
&= 1.44
\end{aligned}
$$

Q 8 From the following table compute (a) $E(X)$ **(b)** $E(2X \pm 3)$ **(c)** $E(24X + 5)$ **(d)** $E(X^2)$ **(e)** $V(X)$ [?]

X	-3	-2	-1	0	1	2	3
P(X)	0.05	0.1	0.3	0	0.3	0.15	0.1

Solution:

Mean

$$
\begin{aligned}
E[X] &= \sum_{i=1}^{n} x_i p(x_i) \\
&= -3 \times 0.05 - 2 \times 0.1 - 1 \times 0.3 + 0 \times 0 + 1 \times 0.3 + 2 \times 0.153 \times 0.1 \\
&= 0.25
\end{aligned}
$$

(b) $E(2X \pm 3)$

$$
\begin{aligned}
E(2X \pm 3) &= 2E(X) \pm 3 \\
&= 2 \times 0.25 \pm 3 \\
&= 0.5 \pm 3 \\
&= 3.5, \quad -2.5
\end{aligned}
$$

(c) $E(4X + 5)$

$$
\begin{aligned}
E(4X + 5) &= 4E(X) + 5 \\
&= 4 \times 0.25 + 5 \\
&= 6
\end{aligned}
$$

(d) $E(X^2)$

$$
\begin{aligned}
[E(X)]^2 &= (-3)^2(0.05) + (-2)^2(0.1) + (-1)^2(0.3) + (0)^2(0) + (1)^2(0.3) + (2)^2(0.15) + (3)^2(0.1) + \\
&= 2.95
\end{aligned}
$$

(e) $Var(X)$

$$
\begin{aligned}
V[X] &= E(X^2) - [[E(X)]^2] \\
&= 2.95 - (0.25)^2 \\
&= 2.8875
\end{aligned}
$$

Q 29 The pdf for the random variable Y is given as

$$f_Y(y)\begin{cases} 1.5(1-y^2) & 0<y<1 \\ 0 & otherwise \end{cases}$$

What are (a) mean (b) the mean of the square and (c) the variance of the random variable Y? [?]

Solution:

(a)

$$\begin{aligned} \mu_Y &= \int_0^1 y\times 1.5(1-y^2)dy \\ &= 1.5\int_0^1 \left[y-y^3\right]dy \\ &= 1.5\left[\frac{y^2}{2}-\frac{y^4}{4}\right]_0^1 \\ &= 1.5\left\{\left[\frac{1^2}{2}-\frac{1^4}{4}\right]-[0]\right\} \\ &= 0.3750 \end{aligned}$$

(b)

$$\begin{aligned} \overline{Y^2} &= \int_0^1 y^2\times 1.5(1-y^2)dy \\ &= 1.5\int_0^1 \left[y^2-y^4\right]dy \\ &= 1.5\left[\frac{y^3}{3}-\frac{y^5}{5}\right]_0^1 \\ &= 1.5\left\{\left[\frac{1^3}{3}-\frac{1^5}{5}\right]-[0]\right\} \\ &= 0.2 \end{aligned}$$

(c)

$$\begin{aligned} \sigma_Y^2 &= \overline{Y^2}-\mu_Y^2 \\ &= 0.2-(0.3750)^2 \\ &= 0.0594 \end{aligned}$$

Q 30 The pdf for the random variable Z is given as

$$f_Z(z)\begin{cases} \frac{1}{6\sqrt{z}} & 0<y<9 \\ 0 & otherwise \end{cases}$$

What are (a) mean (b) the mean of the square and (c) the variance of the random variable Z ? [?]

Solution:

(a)

$$\begin{aligned} \mu_Z &= \int_0^1 z\frac{1}{6\sqrt{z}}dz \\ &= \frac{1}{6}\int_0^9 \left[z^{(1/2)}\right]dz \\ &= \frac{1}{6}\left[\frac{z^{(3/2)}}{(3/2)}\right]_0^9 = \frac{1}{6}\left[\frac{9^{(3/2)}}{(3/2)}\right] \\ &= \frac{1}{9}[27] \\ &= 3 \end{aligned}$$

(b)

$$\begin{aligned} \overline{Z^2} &= \int_0^1 z^2\frac{1}{6\sqrt{z}}dz \\ &= \frac{1}{6}\int_0^9 \left[z^{(3/2)}\right]dz \\ &= \frac{1}{6}\left[\frac{z^{(5/2)}}{(5/2)}\right]_0^9 \frac{1}{6}\left[\frac{9^{(5/2)}}{(5/2)}\right]_0^9 \\ &= \frac{1}{15}[243] \\ &= 16.2 \end{aligned}$$

(c)

$$\begin{aligned} \sigma_Z^2 &= \overline{Z^2}-\mu_Z^2 \\ &= 16.2-(3)^2 \\ &= 7.2000 \end{aligned}$$

Q 31 The pdf for the random variable X is given as

$$f_X(x)\begin{cases} 0.5303\sqrt{x} & 0<x<2 \\ 0 & otherwise \end{cases}$$

What are (a) mean (b) the mean of the square and (c) the variance of the random variable X ? [?]

Solution:

(a)

$$\begin{aligned} \mu_X &= \int_0^2 x\times 0.5303\sqrt{x}dx \\ &= 0.5303\int_0^2 \left[x^{(3/2)}\right]dx \\ &= 0.5303\left[\frac{x^{(5/2)}}{(5/2)}\right]_0^2 = 0.5303\left[\frac{2^{(5/2)}}{(5/2)}\right] \\ &= 0.5303\left[\frac{5.6566}{2.5}\right] \\ &= 1.1999 \end{aligned}$$

(b)

$$\begin{aligned} \overline{X^2} &= \int_0^2 x^2\times 0.5303\sqrt{x}dx \\ &= 0.5303\int_0^2 \left[x^{(5/2)}\right]dx \\ &= 0.5303\left[\frac{x^{(7/2)}}{(7/2)}\right]_0^2 = 0.5303\left[\frac{2^{(7/2)}}{(7/2)}\right] \\ &= 0.5303\left[\frac{11.3137}{3.5}\right] \\ &= 1.7142 \end{aligned}$$

(c)

$$\begin{aligned} \sigma_X^2 &= \overline{Y^2}-\mu_Y^2 \\ &= 1.7142-(1.1999)^2 \\ &= 0.2744 \end{aligned}$$

Q 32 A pdf is described by $\frac{1}{8}(z-6)$ for all values of z between 6 and 10 and is 0 otherwise. What are (a) the mean (b) the mean of the square and (c) the variance of the random variable Z ? [?]

Solution:

$$\begin{aligned} \mu_Z &= \frac{1}{8}\int_6^{10} z\times(z-6)dz \\ &= \frac{1}{8}\int_6^{10}(z^2-6z)dz \\ &= \frac{1}{8}\left[\frac{z^3}{3}-6\frac{z^2}{2}\right]_6^{10} \\ &= \frac{1}{8}\left[\left(\frac{10^3}{3}-6\frac{10^2}{2}\right)-\left(\frac{6^3}{3}-6\frac{6^2}{2}\right)\right] \\ &= \frac{1}{8}[(333.33-300)-(72-108)] \\ &= \frac{1}{8}[33.333+36] \\ &= 8.6666 \end{aligned}$$

(b)

$$\begin{aligned}
\overline{Z^2} &= \frac{1}{8}\int_6^{10} z^2 \times (z-6)dz \\
&= \frac{1}{8}\int_6^{10} (z^3 - 6z^2)dz \\
&= \frac{1}{8}\left[\frac{z^4}{4} - 6\frac{z^3}{3}\right]_6^{10} \\
&= \frac{1}{8}\left[\left(\frac{10^4}{4} - 6\frac{10^3}{3}\right) - \left(\frac{6^4}{4} - 6\frac{6^3}{3}\right)\right] \\
&= \frac{1}{8}\left[(2500 - 2000) - (324 - 432)\right] \\
&= \frac{1}{8}\left[500 + 108\right] \\
&= 76
\end{aligned}$$

(c)

$$\begin{aligned}
\sigma_Z^2 &= \overline{Z^2} - \mu_Z^2 \\
&= 76 - (8.6666)^2 \\
&= 0.8889
\end{aligned}$$

Q 33 A pdf is described by $0.05(x-3)$ for all values of x between 6 and 10 and is 0 otherwise. What are (a) the mean (b) the mean of the square and (c) the variance of the random variable X ? [?]

Solution:

$$\begin{aligned}
\mu_X &= \int_6^{10} x \times 0.05(x-3)dx \\
&= 0.05\int_6^{10} (x^2 - 3x)dx \\
&= 0.05\left[\frac{x^3}{3} - 3\frac{x^2}{2}\right]_6^{10} \\
&= 0.05\left[\left(\frac{10^3}{3} - 3\frac{10^2}{2}\right) - \left(\frac{6^3}{3} - 3\frac{6^2}{2}\right)\right] \\
&= 0.05\left[(333.33 - 150) - (72 - 54)\right] \\
&= 0.05\left[183.333 - 18\right] \\
&= 8.2666
\end{aligned}$$

(b)

$$\begin{aligned}
\overline{X^2} &= 0.05\int_6^{10} x^2 \times (x-3)dx \\
&= 0.05\int_6^{10} (x^3 - 3x^2)dx \\
&= 0.05\left[\frac{x^4}{4} - 3\frac{x^3}{3}\right]_6^{10} \\
&= 0.05\left[\left(\frac{10^4}{4} - 3\frac{10^3}{3}\right) - \left(\frac{6^4}{4} - 3\frac{6^3}{3}\right)\right] \\
&= 0.05\left[(2500 - 1000) - (324 - 216)\right] \\
&= 0.05\left[1500 - 108\right] \\
&= 69.6
\end{aligned}$$

(c)

$$\begin{aligned}
\sigma_X^2 &= \overline{Z^2} - \mu_Z^2 \\
&= 69.6 - (8.2666)^2 \\
&= 1.2633
\end{aligned}$$

Q 34 A pdf is described by $\frac{1}{16}(y-4)$ for all values of y between 6 and 10 and is 0 otherwise. What are (a) the mean (b) the mean of the square and (c) the variance of the random variable Y ? [?]

Solution:

(a)

$$\begin{aligned}
\mu_Y &= \frac{1}{16}\int_6^{10} y \times (y-4)dy \\
&= \frac{1}{16}\int_6^{10} (y^2 - 4y)dy \\
&= \frac{1}{16}\left[\frac{y^3}{3} - 4\frac{y^2}{2}\right]_6^{10} \\
&= \frac{1}{16}\left[\left(\frac{10^3}{3} - 4\frac{10^2}{2}\right) - \left(\frac{6^3}{3} - 4\frac{6^2}{2}\right)\right] \\
&= \frac{1}{16}\left[(333.33 - 200) - (72 - 72)\right] \\
&= \frac{1}{16}\left[133.333\right] \\
&= 8.3333
\end{aligned}$$

(b)

$$\begin{aligned}
\overline{Y^2} &= \frac{1}{8}\int_6^{10} y^2 \times (y-4)dy \\
&= \frac{1}{16}\int_6^{10} (y^3 - 4y^2)dz \\
&= \frac{1}{16}\left[\frac{y^4}{4} - 4\frac{y^3}{3}\right]_6^{10} \\
&= \frac{1}{16}\left[\left(\frac{10^4}{4} - 4\frac{10^3}{3}\right) - \left(\frac{6^4}{4} - 4\frac{6^3}{3}\right)\right] \\
&= \frac{1}{16}\left[(2500 - 1333.333) - (324 - 288)\right] \\
&= \frac{1}{16}\left[1166.67 - 36\right] \\
&= 70.666
\end{aligned}$$

(c)

$$\begin{aligned}
\sigma_Z^2 &= \overline{Z^2} - \mu_Z^2 \\
&= 70.666 - (8.3333)^2 \\
&= 1.22711
\end{aligned}$$

Q 35 It is given that $E[X] = 2.0$ and $E[X^2] = 6.0$ Find (a) the standard deviation of X (b) If $Y = 6X^2 + 2X - 13$ find μ_Y [?]

Solution:

(a) Standard deviation σ is

$$\begin{aligned}
\sigma_X^2 &= E[X^2] - \mu_X^2 \\
&= 6 - (2)^2 \\
&= 2 \\
\sigma &= 1.4142
\end{aligned}$$

(b)

$$\begin{aligned}
E[Y] &= 6E[X^2] + 2E[X] - 13 \\
&= 6 \times 6 + 2 \times 2 - 13 \\
&= 27
\end{aligned}$$

Q 36 It is given that $E[X] = 36.5$ and $E[X^2] = 1432.3$ Find (a) the standard deviation of X (b) If $Y = 4X - 500$ find the mean and variance of Y [?]

Solution:

(a) Standard deviation σ is

$$\begin{aligned}
\sigma_X^2 &= E[X^2] - \mu_X^2 \\
&= 1432.3 - (36.5)^2 \\
&= 100.05 \\
\sigma &= 10.0025
\end{aligned}$$

(b)

$$\begin{aligned} \mu_Y &= 4E[X] - 500 \\ &= 4 \times 36.5 - 500 \\ &= -354 \end{aligned}$$

$$\begin{aligned} \sigma^2 Y &= 4^2 \sigma_X^2 \\ &= 16 \times 100.05 \\ &= 1600.8 \end{aligned}$$

Q 37 It is given that $E[X] = 13.5$ and $E[X^2] = 1578.6$ (a) Find the standard deviation of X (b) If $W = \frac{1}{3}X^2$ find $E[W]$ [?]

Solution:

(a) Standard deviation σ is

$$\begin{aligned} \sigma_X^2 &= E[X^2] - \mu_X^2 \\ &= 1578.6 - (13.5)^2 \\ &= 1396.35 \\ \sigma &= 37.3678 \end{aligned}$$

(b)

$$\begin{aligned} E[W] &= \frac{1}{3}E[X^2] \\ &= \frac{1}{3} \times 1578.6 \\ &= 326.2 \end{aligned}$$

Q 38 It is given that random variable X has the expectations $\mu_X = 12.7$ and $E[X^2] = 313.3$. The random variable Y is obtained from X using $Y = -2.7X + 16$. (a) Find σ_X (b) Find the mean and the variance of Y [?]

Solution:

(a) Standard deviation σ is

$$\begin{aligned} \sigma_X^2 &= E[X^2] - \mu_X^2 \\ &= 313.3 - (12.7)^2 \\ &= 152.01 \\ \sigma_X &= 12.3292 \end{aligned}$$

(b)

$$\begin{aligned} \mu_Y &= -2.7\mu_X + 16 \\ &= -2.7 \times 12.7 + 16 \\ &= -18.2900 \\ \sigma_Y^2 &= (-2.7)^2 sigma_X^2 \\ &= (-2.7)^2 \times 152.01 \\ &= 1108.1599 \end{aligned}$$

Q 39 The random variable X has the expectations $\mu_X = -24$ and $\sigma_X = 3$. The random variable Y is obtained from X using $Y = -2.7X + 16$. (a) Find $E[X^2]$ (b) Find the mean and the variance of Y [?]

Solution:

(a) $E[X^2]$ is

$$\begin{aligned} E[X^2] &= \sigma_X^2 + \mu_X^2 \\ &= (3)^2 + (-24)^2 \\ &= 585 \end{aligned}$$

(b)

$$\begin{aligned} \mu_Y &= -2.7\mu_X + 16 \\ &= -2.7 \times (-24) + 16 \\ &= 80.8 \\ \sigma_Y^2 &= (-2.7)^2 sigma_X^2 \\ &= (-2.7)^2 \times (3)^2 \\ &= 65.61 \end{aligned}$$

Q 40 The random variable X has the expectations $\mu_X = 37.9$ and $\sigma_X = 10.3$. The random variable Y is obtained from X using $Y = 0.6X^2 + 7.9X + 107.1$. (a) Find $E[X^2]$ (b) Find the mean of Y [?]

Solution:

(a) $E[X^2]$ is

$$\begin{aligned} E[X^2] &= \sigma_X^2 + \mu_X^2 \\ &= (10.3)^2 + (37.9)^2 \\ &= 1542.5 \end{aligned}$$

(b)

$$\begin{aligned} \mu_Y &= 0.6E[X^2] + 7.9E[X] + 107.1 \\ &= 0.6 \times (37.9) + 7.9 \times 37.9 + 107.1 \\ &= 1332.01 \end{aligned}$$

Exercise:47 Given the data in the following table, a) What are the mean and variance of Y b) If $W = Y^2 + 1$. What are the mean and variance of W? [?]

$k = i$	1	2	3	4	5
y_k	2.1	3.2	4.8	5.4	6.9
$P(y_k)$	0.20	0.21	0.19	0.14	0.26

Solution:

$k = i$	1	2	3	4	5
y_k	2.1	3.2	4.8	5.4	6.9
$P(y_k)$	0.20	0.21	0.19	0.14	0.26
$w_k = Y^2 + 1$	5.41	11.24	24.04	30.16	48.61

a) Mean

$$\mu_Y = y_k P(y_k) = 2.1(0.20) + 3.2(0.21) + 4.8(0.19) + 5.4(0.14) + 6.9(0.26) = 4.5540$$

$$
\begin{aligned}
Y^2 &= y_k^2 P(y_k) \\
&= (2.1)^2(0.20) + (3.2)^2(0.21) + (4.8)^2(0.19) + (5.4)^2(0.14) + (6.9)^2(0.26) \\
&= 23.8710
\end{aligned}
$$

$$\sigma_Y^2 = Var(Y) \quad = \quad Y^2 - \mu_Y^2 = 23.8710 - (4.5540)^2 = 3.1321$$

b) Mean

$$\mu_W \quad = \quad w_k P(y_k) = 5.41(0.20) + 11.24(0.21) + 24.04(0.19) + 30.16(0.14) + 48.61(0.26) = 24.8710$$

$$
\begin{aligned}
WY^2 &= w_k^2 P(y_k) \\
&= (5.41)^2(0.20) + (11.24)^2(0.21) + (24.04)^2(0.19) + (30.16)^2(0.14) + (48.61)^2(0.26) \\
&= 883.8996
\end{aligned}
$$

$$\sigma_W^2 = Var(W) \quad = \quad W^2 - \mu_W^2 = 883.8996 - (24.8710)^2 = 265.3329$$

Exercise:48 Given the data in the following table, a) What are the mean and variance of X b) If $Y = X^2 + 2$. What are the μ_Y and σ_Y^2 ? [?]

$k = i$	1	2	3	4	5
x_k	2.1	3.2	4.8	5.4	6.9
$P(x_k)$	0.21	0.18	0.20	0.22	0.19

Solution:

$k = i$	1	2	3	4	5
x_k	2.1	3.2	4.8	5.4	6.9
$P(x_k)$	0.21	0.18	0.20	0.22	0.19
$y_k = x^2 + 2$	6.41	12.24	25.04	31.16	49.61

a) Mean

$$\mu_X \quad = \quad x_k P(x_k) = 2.1(0.21) + 3.2(0.18) + 4.8(0.20) + 5.4(0.22) + 6.9(0.19) = 4.4760$$

$$
\begin{aligned}
X^2 &= x_k^2 P(x_k) \\
&= (2.1)^2(0.21) + (3.2)^2(0.18) + (4.8)^2(0.20) + (5.4)^2(0.22) + (6.9)^2(0.19) \\
&= 22.8384
\end{aligned}
$$

$$\sigma_X^2 = Var(X) \quad = \quad X^2 - \mu_X^2 = 22.8384 - (4.4760)^2 = 2.8038$$

b) Mean

$$\mu_Y \quad = \quad y_k P(x_k) = 6.41(0.21) + 12.24(0.18) + 25.04(0.20) + 31.16(0.22) + 49.61(0.19) = 24.8384$$

$$
\begin{aligned}
Y^2 &= y_k^2 P(x_k) \\
&= (6.41)^2(0.21) + (12.24)^2(0.18) + (25.04)^2(0.20) + (31.16)^2(0.22) + (+49.61)^2(0.19) \\
&= 842.2229
\end{aligned}
$$

$$\sigma_Y^2 = Var(Y) \quad = \quad Y^2 - \mu_Y^2 = 842.2229 - (24.8384)^2 = 225.2768$$

Exercise:49 Given the data in the following table, a) What are the mean and variance of Z b) If $U = X^2 + 3$. What are the μ_U and σ_U^2 ? [?]

Solution:

a) Mean

$$\mu_Z \quad = \quad z_k P(z_k) = 2.1(0.19) + 3.2(0.22) + 4.8(0.20) + 5.4(0.18) + 6.9(0.21) = 4.4840$$

$k=i$	1	2	3	4	5
z_k	2.1	3.2	4.8	5.4	6.9
$P(z_k)$	0.19	0.22	0.20	0.18	0.21

$k=i$	1	2	3	4	5
z_k	2.1	3.2	4.8	5.4	6.9
$P(z_k)$	0.19	0.22	0.20	0.18	0.21
$u_k = z^2+3$	7.41	13.24	26.04	32.16	50.61

$$\begin{aligned} Z^2 &= z_k^2 P(z_k) \\ &= (2.1)^2(0.19) + (3.2)^2(0.22) + (4.8)^2(0.20) + (5.4)^2(0.18) + (6.9)^2(0.21) \\ &= 22.9456 \end{aligned}$$

$$\sigma_Z^2 = Var(Z) \quad = \quad Z^2 - \mu_Z^2 = 22.9456 - (4.4840)^2 = 2.8393$$

b) Mean

$$\mu_U \quad = \quad u_k P(z_k) = 7.41(0.19) + 13.24(0.22) + 26.04(0.20) + 32.16(0.18) + 50.61(0.21) = 25.9456$$

$$\begin{aligned} U^2 &= u_k^2 P(y_k) \\ &= (7.41)^2(0.19) + (13.24)^2(0.22) + (26.04)^2(0.20) + (32.16)^2(0.18) + (50.61)^2(0.21) \\ &= 908.6703 \end{aligned}$$

$$\sigma_U^2 = Var(U) \quad = \quad U^2 - \mu_U^2 = 908.6703 - (25.9456)^2 = 235.4961$$

1.10 Characteristic Function:[?, ?, ?]

Characteristic Function:

Learning Outcomes
On completion the students are able to

- understand the characteristic function.
- understand the characteristic function for different distribution functions.
- estimating the characteristic function for different functions.

Introduction

- In some of the distributions mean and variance mathematically does not exist.
- The characteristic function can be more general alternative description of pdf than mean, variance and the higher order moments.
- The characteristic function is more attractive in advanced probability topics.
- In this situation a new function defined which is called as characteristic function.
- It is similar to Fourier transform, which is more useful.

Consider a random variable X and its pdf is defined as $f(x)$ then its characteristic function is defined is given by

$$\phi_X(j\omega) = E[e^{j\omega x}] = \int_{-\infty}^{\infty} f_X(x) e^{j\omega x} dx$$

The characteristic function and the pdf are Fourier transforms pair:

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_X(j\omega) e^{-j\omega x} d\omega$$

Moments (Mean μ_X and Variance σ_X^2)about the origin may be obtained from a characteristic function by the following procedure.

$$\frac{d^n}{d(\omega)^n} \phi_X(j\omega) = \int_{-\infty}^{\infty} x^n f_X(x) e^{j\omega x} dx$$

$$E[X] = \frac{d}{d(\omega)} \phi_X(j\omega)|_{\omega=0} = \mu_X$$

$$E[X^2] = \frac{d^2}{d(\omega)^2} \phi_X(j\omega)|_{\omega=0} = \mu_X^2$$

Variance is

$$\sigma_X^2 = E[X^2] - E[X]^2$$

Example 1. Estimate the characteristic function for a random variable X which is uniformly distributed between 0 and 1. [?]

Solution:

$$\begin{aligned}
\phi_X(j\omega) &= \int_0^1 f_X(x)e^{j\omega x}dx = \int_0^1 1e^{j\omega x}dx \\
&= \left[\frac{e^{j\omega x}}{e^{j\omega}}\right]_0^1 \\
&= \frac{e^{j\omega-1}}{j\omega}
\end{aligned}$$

Example 2. Estimate the characteristic function for a normalized Gaussian random variable. [?]

Solution:

$$f_Z(z) = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}$$

$$\begin{aligned}
\phi_Z(j\omega) &= \int_{-\infty}^{\infty} f_Z(z)e^{j\omega z}dz \\
&= \int_{-\infty}^{1} \frac{\infty}{\sqrt{2\pi}}e^{-z^2}e^{j\omega z}dz \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{(-z^2/2+j\omega z)}dz \\
&= \frac{1}{\sqrt{2\pi}}e^{-\omega^2/2}\int_{-\infty}^{\infty} e^{-(z-j\omega)^2/2}dz
\end{aligned}$$

$$u = z - j\omega$$

$$\begin{aligned}
\phi_Z(j\omega) &= \frac{1}{\sqrt{2\pi}}e^{-\omega^2/2}\int_{-\infty}^{\infty} e^{-u^2/2}du \\
\phi_Z(j\omega) &= e^{-\omega^2/2}
\end{aligned}$$

Example 3. Estimate the characteristic function for pdf $f_X(x) = \sum_{i=1}^{n} P_i\delta(x-x_i)$. [?]

Solution:

$$\begin{aligned}
\phi_X(j\omega) &= \int_{-\infty}^{\infty} f_X(x)e^{j\omega x}dx \\
&= \int_{-\infty}^{\infty}\sum_{i=1}^{n} P_i\delta(x-x_i)e^{j\omega x}dx \\
&= \sum_{i=1}^{n} P_i int_{-\infty}^{\infty}\delta(x-x_i)e^{j\omega x}dx \\
&= \sum_{i=1}^{n} P_i e^{j\omega x_i}
\end{aligned}$$

Example 3. Estimate the characteristic function of Bernoulli counting random variable pdf $f_X(x) = q\delta(x) + p\delta(x-1)$. [?]

Solution:

$$\begin{aligned}
\phi_X(j\omega) &= \int_{-\infty}^{\infty} f_X(x)e^{j\omega x}dx \\
&= \int_{-\infty}^{\infty} [q\delta(x) + p\delta(x-1)]e^{j\omega x}dx \\
&= q + pe^{j\omega}
\end{aligned}$$

53. Verify that the characteristic function $\phi_X(j\omega) = \frac{\lambda}{\lambda - j\omega}$ for the exponential random variable. [?]

Solution:

$$\begin{aligned}
\phi_X(j\omega) &= \int_{-\infty}^{\infty} f_X(x)e^{j\omega x}dx \\
&= \int_{-\infty}^{\infty} \lambda e^{-\lambda x}e^{j\omega x}dx \\
&= \lambda\int_0^{\infty} e^{-(\lambda-j\omega)x}dx \\
&= \lambda\left[\frac{e^{-(\lambda-j\omega)x}}{\lambda-j\omega}\right]_0^{\infty} \\
&= \frac{\lambda}{\lambda-j\omega}
\end{aligned}$$

54. Verify that the characteristic function $\phi_X(j\omega) = \frac{b^2}{\omega^2+b^2}$ for the Laplace random variable. [?]

Solution:

$$\begin{aligned}
\phi_X(j\omega) &= \int_{-\infty}^{\infty} f_X(x)e^{j\omega x}dx \\
&= \int_{-\infty}^{\infty} \frac{b}{2}exp(-b|x|)e^{j\omega x}dx \\
&= \frac{b}{2}\int_{-\infty}^{\infty} e^{-bx}e^{j\omega x}dx \\
&= \frac{b}{2}\left[\int_{-\infty}^{0} e^{(b+j\omega)x} + \int_0^{\infty} e^{-(b-j\omega)x}\right] \\
&= \frac{b}{2}\left[\frac{1}{b+j\omega} + \frac{1}{b-j\omega}\right] \\
&= \frac{b^2}{\omega^2+b^2}
\end{aligned}$$

56. Verify that the characteristic function $\phi_X(j\omega) = exp(-a|\omega|)$ for the Cauchy random variable. [?]

Solution:

$$\begin{aligned}
\phi_X(j\omega) &= \int_{-\infty}^{\infty} f_X(x)e^{j\omega x}dx \\
&= \int_{-\infty}^{\infty} \frac{a}{\pi(x^2+a^2)}e^{j\omega x}dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-a|\omega|}e^{-j\omega x}d\omega \\
&= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{(a-jx)\omega}d\omega + \int_0^{\infty} e^{-(a-jx)\omega}\right] \\
&= \frac{1}{2\pi}\left[\frac{1}{a-jx} + \frac{1}{a+jx}\right] \\
&= \frac{a}{2pi}\frac{a}{x^2+a^2}
\end{aligned}$$

57. Use the characteristic function of the normalized random variable, found in (2.86) to find first two moments about the origin $E[Z]$ $E[Z^2]$. With these results find the variance σ_Z^2. [?]

Solution:

$$\begin{aligned}
\phi_Z(j\omega) &= e^{(j\omega)^2/2} \\
\frac{d}{d(j\omega)}\phi_Z(j\omega) &= (1+(j\omega)^2 e^{(j\omega)^2/2} \\
\frac{d^2}{d(j\omega)^2}\phi_Z(j\omega) &= j\omega e^{(j\omega)^2/2} \\
\mu_Z &= \frac{d}{d(j\omega)}\phi_Z(j\omega)|_{\omega=0} \\
&= e_{\omega=0}^{(j\omega)^2/2} = 0 \\
E[Z^2] &= \frac{d^2}{d(j\omega)^2}\phi_Z(j\omega)|_{\omega=0} \\
&= (1+(j\omega)^2 e_{\omega=0}^{(j\omega)^2/2} = 1 \\
\sigma_Z^2 &= E[Z^2] - \mu_Z^2 = 1
\end{aligned}$$

58. Use the characteristic function of the Laplace random variable, found in Appendix A to find first two moments about the origin $E[X]$ $E[X^2]$. With these results find the variance σ_X^2. [?]

Solution:

$$\begin{aligned}
\phi_X(j\omega) &= \frac{b^2}{b^2-(j\omega)^2} \\
\frac{d}{d(j\omega)}\phi_X(j\omega) &= \frac{2b^2(j\omega)}{(b^2-(j\omega)^2)^2} \\
\frac{d^2}{d(j\omega)^2}\phi_X(j\omega) &= \frac{2b^2(b^2-(j\omega)^2) - 2b^2(j\omega)}{(b^2-(j\omega)^2)^4} \\
\mu_X &= \frac{d}{d(j\omega)}\phi_X(j\omega)|_{\omega=0} \\
&= \frac{2b^2(j\omega)}{(b^2-(j\omega)^2)^2}|_{\omega=0} = 0 \\
E[X^2] &= \frac{d^2}{d(j\omega)^2}\phi_X(j\omega)|_{\omega=0} \\
&= \left.\frac{2b^2(b^2-(j\omega)^2) - 2b^2(j\omega)}{(b^2-(j\omega)^2)^4}\right|_{\omega=0} = 2/b^2 \\
\sigma_X^2 &= E[X^2] - \mu_X^2 = 2/b^2
\end{aligned}$$

59. Use the characteristic function of the gamma random variable, found in Appendix A to find first two moments about the origin $E[X]$ $E[X^2]$. With these results find the variance σ_X^2. [?]

Solution:

$$\begin{aligned}
\phi_X(j\omega) &= (1-j\omega b)^{(-a)} \\
\frac{d}{d(j\omega)}\phi_X(j\omega) &= ab(1-j\omega b)^{(-a-1)} \\
\frac{d^2}{d(j\omega)^2}\phi_X(j\omega) &= (a+1)ab^2(1-j\omega b)^{(-a-2)} \\
\mu_X &= \frac{d}{d(j\omega)}\phi_X(j\omega)|_{\omega=0} \\
&= ab(1-j\omega b)^{(}-a-1)_{\omega=0} = ab \\
E[X^2] &= \frac{d^2}{d(j\omega)^2}\phi_X(j\omega)|_{\omega=0} \\
&= (a+1)ab^2(1-j\omega b)^{(-a-2)}|_{\omega=0} \\
&= ab^2(a+1) \\
\sigma_X^2 &= E[X^2] - \mu_X^2 = ab^2
\end{aligned}$$

Note: Entire material is taken from different text books or from the Internet (different websites). Slightly it is modified from the original content. It is not for any commercial purpose. It is used to teach students. Suggestions are always encouraged.

1.11 Conditioned Random Variables:[?, ?, ?]

88. Given the discrete random variable

$$f_X(x) = 0.25\delta(x) + 0.25\delta(x-1) + 0.25\delta(x-2) + 0.25\delta(x-3)$$

The event B ia $B = \{X > 0\}$ **What are the pdf and cdf conditioned by event B?** [?]

Solution:

$$f_X(x) = 0.25\delta(x) + 0.25\delta(x-1) + 0.25\delta(x-2) + 0.25\delta(x-3)$$

$$B = \{X > 0\}$$

$$\begin{aligned} P(B) &= P(X=1) + P(X=2) + P(X=3) \\ &= 0.25 + 0.25 + 0.25 = 0.75 \end{aligned}$$

$$\begin{aligned} f_{X|B}(x) &= \frac{f_X(x)}{P(B)} \\ &= \frac{0.25}{0.75}\delta(x-1) + \frac{0.25}{0.75}\delta(x-2) + \frac{0.25}{0.75}\delta(x-3) \\ &= \frac{1}{3}\delta(x-1) + \frac{1}{3}\delta(x-2) + \frac{1}{3}\delta(x-3) \\ F_{X|B}(x) &= \frac{1}{3}u(x-1) + \frac{1}{3}u(x-2) + \frac{1}{3}u(x-3) \end{aligned}$$

89. Given the discrete random variable

$$f_X(x) = 0.37\delta(x) + 0.16\delta(x-1) + 0.29\delta(x-2) + 0.18\delta(x-3)$$

The event B ia $B = \{X > 1\}$ **What are the pdf and cdf conditioned by event B?** [?]

Solution:

$$f_X(x) = 0.37\delta(x) + 0.16\delta(x-1) + 0.29\delta(x-2) + 0.18\delta(x-3)$$

$$B = \{X > 1\}$$

$$\begin{aligned} P(B) &= P(X=2) + P(X=3) \\ &= 0.29 + 0.18 = 0.47 \end{aligned}$$

$$\begin{aligned} f_{X|B}(x) &= \frac{f_X(x)}{P(B)} \\ &= \frac{0.29}{0.47}\delta(x-2) + \frac{0.18}{0.47}\delta(x-3) \\ &= 0.617\delta(x-2) + 0.383\delta(x-3) \\ F_{X|B}(x) &= 0.617u(x-2) + 0.383u(x-3) \end{aligned}$$

90. Given the discrete random variable

$$f_X(x) = 0.37\delta(x) + 0.16\delta(x-1) + 0.29\delta(x-2) + 0.18\delta(x-3)$$

The event B ia $B = \{X \leq 1\}$ **What are the pdf and cdf conditioned by event B?** [?]

Solution:

$$f_X(x) = 0.37\delta(x) + 0.16\delta(x-1) + 0.29\delta(x-2) + 0.18\delta(x-3)$$

$$B = \{X \leq 1\}$$

$$\begin{aligned} P(B) &= P(X=0) + P(X=1) \\ &= 0.37 + 0.16 = 0.53 \end{aligned}$$

$$
\begin{aligned}
f_{X|B}(x) &= \frac{f_X(x)}{P(B)} \\
&= \frac{0.37}{0.53}\delta(x) + \frac{0.16}{0.53}\delta(x-1) \\
&= 0.6981\delta(x) + 0.3019\delta(x-1) \\
F_{X|B}(x) &= 0.6981u(x) + 0.3019u(x-1)
\end{aligned}
$$

91. The random variable X is uniformly distributed between 0 and 5. The event B is $B = \{X > 3.7\}$ What are $f_{X|B}(x)$, $\mu_{X|B}(x)$ and $\sigma^2_{X|B}(x)$? [?]

Solution:

The random variable X is uniformly distributed between 0 and 5. Therefore its pdf is

$$f_X(x) = \frac{1}{5} = 0.2$$

$$B = \{X > 3.7\}$$

$$P(B) = \frac{5-3.7}{5} = 0.2600$$

$$
\begin{aligned}
f_{X|B}(x) &= \frac{f_X(x)}{P(B)} \\
&= \frac{0.2}{0.26} = 0.7692 \quad 3.7 < x < 5 \\
\mu_{X|B} &= \frac{5+3.7}{2} = 4.35 \\
\sigma^2_{X|B}(x) &= \frac{(5-3.7)^2}{12} = 0.1408
\end{aligned}
$$

92. The random variable X is uniformly distributed between 1 and 6. The event B is $B = \{X > 3.7\}$ What are the pdf, the mean and the variance of the random variable X conditioned by the even B? [?]

Solution:

The random variable X is uniformly distributed between 1 and 6. Therefore its pdf is

$$f_X(x) = \frac{1}{6-1} = 0.2$$

$$B = \{X > 3.7\}$$

$$P(B) = \frac{6-3.7}{5} = 0.46$$

$$
\begin{aligned}
f_{X|B}(x) &= \frac{f_X(x)}{P(B)} \\
&= \frac{0.2}{0.46} = 0.4348 \quad 3.7 < x < 6 \\
\mu_{X|B} &= \frac{6+3.7}{2} = 4.851 \\
\sigma^2_{X|B}(x) &= \frac{(6-3.7)^2}{12} = 0.4408
\end{aligned}
$$

93. The random variable X is uniformly distributed between 2 and 7. The event B is $B = \{X > 3.7\}$ What are the pdf, the mean and the variance of the random variable X conditioned by the even B? [?]

Solution:

The random variable X is uniformly distributed between 0 and 5. Therefore its pdf is

$$f_X(x) = \frac{1}{7-2} = 0.2$$

$$B = \{X > 3.7\}$$

$$P(B) = \frac{7-3.7}{5} = 0.66$$

$$\begin{aligned}
f_{X|B}(x) &= \frac{f_X(x)}{P(B)} \\
&= \frac{0.2}{0.66} = 0.3030 \quad 3.7 < x < 7 \\
\mu_{X|B} &= \frac{7+3.7}{2} = 5.35 \\
\sigma^2_{X|B}(x) &= \frac{(7-3.7)^2}{12} = 0.9075
\end{aligned}$$

94. Let X be an exponential random variable

$$F_X(x)\begin{cases} 1-\exp(-x/3) & x \geq 0 \\ 0 & x < 0 \end{cases}$$

and let B be the event $B = \{X > 2\}$ What are $f_{X|B}(x)$, $\mu_{X|B}(x)$ and $\sigma^2_{X|B}$? [?]

Solution:

$$B = \{X > 2\}$$

$$\begin{aligned}
P(B) &= 1 - P(X \leq 2) = 1 - F_X(2) \\
&= 1 - (1 - \exp(-x/3)) = exp(-x/3) \\
&= exp(-2/3) = 0.5134
\end{aligned}$$

$$\begin{aligned}
f_X(x) &= \frac{d}{dx}[F(x)] \\
&= \frac{d}{dx}[1 - e^{(-x/3)}]dx \\
&= 0 - e^{(-x/3)}(-1/3) \\
&= 1/3e^{(-x/3)}
\end{aligned}$$

$$\begin{aligned}
f_{X|B}(x) &= \frac{f_X(x)}{P(B)} \\
&= \frac{1/3\exp(-x/3)}{0.5314} = 0.6492e^{(-x/3)} \quad x > 2 \\
\mu_{X|B} &= 0.6492\int_2^\infty xe^{(-x/3)}dx = 4.9997 \\
E[X^2|B] &= 0.6492\int_2^\infty x^2e^{(-x/3)}dx = 33.9977 \\
\sigma^2_{X|B}(x) &= E[X^2|B] - \mu^2_{X|B} = 33.9977 - (4.9997) = 9.0007
\end{aligned}$$

95 Let X be an exponential random variable

$$F_X(x)\begin{cases} 1-\exp(-x/4) & x \geq 0 \\ 0 & x < 0 \end{cases}$$

and let B be the event $B = \{X > 2\}$ What are $f_{X|B}(x)$, $\mu_{X|B}(x)$ and $\sigma^2_{X|B}$? [?]

Solution:

$$B = \{X > 2\}$$

$$\begin{aligned}
P(B) &= 1 - P(X \leq 2) = 1 - F_X(2) \\
&= 1 - (1 - \exp(-x/4)) = exp(-x/4) \\
&= exp(-2/4) = 0.6065
\end{aligned}$$

$$\begin{aligned}
f_X(x) &= \frac{d}{dx}[F(x)] \\
&= \frac{d}{dx}[1 - e^{(-x/4)}]dx \\
&= 0 - e^{(-x/4)}(-1/4) \\
&= 1/4e^{(-x/3)}
\end{aligned}$$

$$\begin{aligned}
f_{X|B}(x) &= \frac{f_X(x)}{P(B)} \\
&= \frac{1/4\exp(-x/4)}{0.6065} = 0.4122e^{(-x/4)} \quad x > 2 \\
\mu_{X|B} &= 0.4122\int_2^{\infty} xe^{(-x/4)}dx = 6.0003 \\
E[X^2|B] &= 0.4122\int_2^{\infty} x^2e^{(-x/4)}dx = 52.0025 \\
\sigma^2_{X|B}(x) &= E[X^2|B] - \mu^2_{X|B} = 52.0025 - (6.0003) = 15.9989
\end{aligned}$$

96 Let X be an exponential random variable

$$F_X(x)\begin{cases} 1-\exp(-x/5) & x \geq 0 \\ 0 & x < 0 \end{cases}$$

and let B be the event $B = \{X > 3\}$ What are $f_{X|B}(x)$, $\mu_{X|B}(x)$ and $\sigma^2_{X|B}$? [?]

Solution:

$$B = \{X > 3\}$$

$$\begin{aligned}
P(B) &= 1 - P(X \leq 3) = 1 - F_X(3) \\
&= 1 - (1 - \exp(-x/5)) = exp(-x/5) \\
&= exp(-3/5) = 0.5488
\end{aligned}$$

$$\begin{aligned}
f_X(x) &= \frac{d}{dx}[F(x)] \\
&= \frac{d}{dx}[1 - e^{(-x/5)}]dx \\
&= 0 - e^{(-x/5)}(-1/5) \\
&= 1/5e^{(-x/5)}
\end{aligned}$$

$$\begin{aligned}
f_{X|B}(x) &= \frac{f_X(x)}{P(B)} \\
&= \frac{1/5\exp(-x/5)}{0.5488} = 0.3644e^{(-x/5)} \quad x > 3 \\
\mu_{X|B} &= 0.3644\int_3^{\infty} xe^{(-x/5)}dx = 7.9995 \\
E[X^2|B] &= 0.3644\int_3^{\infty} x^2e^{(-x/5)}dx = 88.9942 \\
\sigma^2_{X|B}(x) &= E[X^2|B] - \mu^2_{X|B} = 88.9942 - (7.9995) = 25.0022
\end{aligned}$$

97 The random variable X is Gaussian with a mean of 997 and standard deviation of 31. What is the probability of B where $B = \{X > 1000\}$ And what is the pdf for X conditioned by B [?]

Solution:

$$B = \{X > 1000\}$$

$$\begin{aligned}
P(B) &= 1 - P(B \leq 1000) = 1 - F_X(1000) \\
&= 1 - \phi[(1000 - 997)/31] = 1 - \phi[0.0968] \\
&= 1 - 0.5385 = 0.4615
\end{aligned}$$

$$\begin{aligned}
f_{X|B}(x) &= \frac{f_X(x)}{P(B)} \\
&= \frac{1}{(0.4615)31\sqrt{2\pi}}exp[-(x-997)^2/2(31)^2] \\
&= 0.0279exp[-(x-997)^2/1992] \quad x > 1000
\end{aligned}$$

Note: Entire material is taken from different text books or from the Internet (different websites). Slightly it is modified from the original content. It is not for any commercial purpose. It is used to teach students. Suggestions are always encouraged.

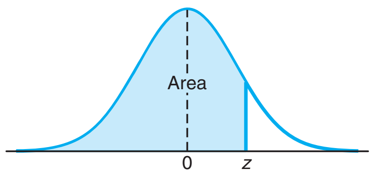

Table A.3 Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

Table A.3 (continued) Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998