Chapter 1

Multiple Random Variables:[?, ?, ?]

1.1 Bivariate-cdf and pdf:

Introduction

- When one measurement is made on each observation, univariate analysis is applied.
- If more than one measurement is made on each observation, multivariate analysis is applied.
- The two measurements will be called X and Y . Since X and Y are obtained for each observation, the data for one observation is the pair (X, Y).

Some examples:

- Height (X) and weight (Y) are measured for each individual in a sample.
- If more than one measurement is made on each observation, multivariate analysis is applied.
- The two measurements will be called X and Y . Since X and Y are obtained for each observation, the data for one observation is the pair (X, Y).
- Temperature (X) and precipitation (Y) are measured on a given day at a set of weather stations.
- The distribution of X and the distribution of Y can be considered individually using univariate methods. That is, we can analyze

$$X_1, X_2, ..., X_n$$

$$Y_1, Y_2, ..., Y_n$$

- using CDFs, densities, quantile functions, etc. Any property that described the behavior of the X_i values alone or the Y_i values alone is called **marginal property.**
- The two measurements will be called X and Y . Since X and Y are obtained for each observation, the data for one observation is the pair (X, Y).
- For example the ECDF $F_X(t)$ of X, the quantile function $Q_Y(p)$ of Y, the sample standard deviation of σ_Y of Y , and the sample mean $\overline{X}$ of X are all marginal properties.
-

Consider a continuous random variables X and Y, then their joint cumulative distribution function (cdf) is defined as:

$$F_{XY}(x, y) = P\{(X \leq x,\ Y \leq y)\}$$

The marginal cdf can be obtained from the joint distribution as:

$$\begin{aligned} F_X(x) &= P(X \leq x,\ Y \leq \infty) = F_{XY}(x, \infty) \\ F_Y(y) &= P(X \leq \infty,\ Y \leq y) = F_{XY}(\infty, y) \end{aligned}$$

Properties of Bivariate Cumulative Density Function (Bivariate cdf)

1. If x and y are very large then the bivariate cdf is

$$F_{XY}(\infty,\infty) = P\{X \leq \infty, Y \leq \infty\} = 1$$

2. The range of cdf is

$$0 \leq F_{XY}(x,y) \leq 1$$

3. The impossible events are

$$\begin{aligned} F_{XY}(-\infty,-\infty) &= P\{X \leq -\infty, Y \leq -\infty\} = 0 \\ F_{XY}(-\infty,y) &= P\{\varnothing(Y \leq y)\} = P(\varnothing) = 0 \\ F_{XY}(x,-\infty) &= 0 \end{aligned}$$

4. Marginal cdfs are

$$\begin{aligned} F_{XY}(\infty,y) &= P\{S \cap (Y \leq y) = F_Y(y) \\ F_{XY}(x,\infty) &= P\{S \cap (X \leq x) = F_X(x) \end{aligned}$$

Independent random variables :

Two random variables$X$ and Y are said to be independent if

$$F_{XY}(x,y) = F_X(x)F_Y(y) \quad for\ all\ x\ and\ y$$

1.1.1 Bivariate Probability Density Function (Bivariate PDF)

Bivariate probability density function (bivariate pdf) is defined as derivative of bivariate cdf and is expressed as

$$f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y) \tag{1.1}$$

The inverse relation of 1.2 is

$$F_{XY}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{XY}(u,v)dudv \tag{1.2}$$

Properties of Bivariate Probability Density Function (Bivariate cdf)

1. The volume of the bivariate pdf is 1 i.e.,

$$f_{XY}(\infty,\infty) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1$$

2. The $F_{XY}(x,y)$ is a non decreasing function

$$f_{XY}(x,y) \geq 0$$

3.

$$P\{x_1 < X \leq x_2,\ y_1 < Y \leq y_2\} = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{XY}(x,y)dxdy$$

4. Marginal pdfs are

$$\begin{aligned} f_X(x) &= \int_{y}^{\infty} f_{XY}(x,y)dy \\ f_Y(y) &= \int_{x}^{\infty} f_{XY}(x,y)dx \end{aligned}$$

Independent random variables :

Two random variables$X$ and Y are said to be independent if

$$
\begin{aligned}
f_{XY}(x,y) &= f_X(x)f_Y(y) \quad for\ all\ x\ and\ y \\
\frac{\partial^2}{\partial x \partial y} F_{XY}(x,y) &= \frac{\partial}{\partial x} f_X(x) \frac{\partial}{\partial y} f_Y(y)
\end{aligned}
$$

3.17. The joint pdf of a bivariate r.v X, Y is given by

$$f_{XY}(x,y) = \begin{cases} k(x+y) & 0<x<2 \quad 0<y<2 \\ 0 & otherwise \end{cases}$$

where k is a constant

[a.] Find the value of k

[b.] Find the marginal pdf's of X and Y

[c.] Are X and Y independent ? [?]

Solution:

a.
It is given that $f_{XY}(x,y) = k(x+y)$ is joint pdf, then

$$\int_{-\infty}^{-\infty}\int_{-\infty}^{-\infty} f(x,y)k(x+y)\ dxdy = 1$$

$$\begin{aligned}
\int_0^2\int_0^2 k(x+y)\ dxdy &= k\int_0^2\left[\int_0^2 (x+y)dx\right]\ dy \\
&= k\int_0^2 [\frac{x^2}{2}+xy]_0^2\ dy \\
1 &= k\int_0^2 (2+2y)\ dy \\
&= k[2y+2\frac{y^2}{2}]_0^2 \\
1 &= 8k \\
k &= \frac{1}{8}
\end{aligned}$$

$$f_{XY}(x,y) = \frac{1}{8}(x+y) \quad 0<x<2 \quad 0<y<2$$

b.

$$\begin{aligned}
f_X(x) &= k\int_0^2 (x+y)\ dy \\
&= k[xy+\frac{y^2}{2}]_0^2 \\
&= k[2x+\frac{4}{2}] = \frac{1}{8}[2x+2] \\
&= \frac{1}{4}[x+1] \quad 0<x<2
\end{aligned}$$

$$\begin{aligned}
f_Y(y) &= k\int_0^2 (x+y)\ dx \\
&= k[\frac{x^2}{2}+xy]_0^2 \\
&= k[\frac{4}{2}+2y] = \frac{1}{8}[2y+2] \\
&= \frac{1}{4}[y+1] \quad 0<y<2
\end{aligned}$$

c.

$$\begin{aligned}
f_X(x)f_Y(y) &= \frac{1}{4}(x+1)\times\frac{1}{4}(y+1) \\
&= \frac{1}{8}(x+1)(y+1)
\end{aligned}$$

$$f_{XY}(x,y) = \frac{1}{8}(x+y) \quad 0<x<2 \quad 0<y<2$$

$$f_{XY}(x,y) \neq f_X(x)f_Y(y)$$

Hence X and Y are not independent

3.18. The joint pdf of a bivariate r.v X, Y is given by

$$f_{XY}(x,y) = \begin{cases} kxy & 0 < x < 1 \quad 0 < y < 1 \\ 0 & otherwise \end{cases}$$

where k is a constant

[a.] Find the value of k.

[b.] Are X and Y independent ?

[c.] Find $P(X+Y<1)$. [?]

Solution:

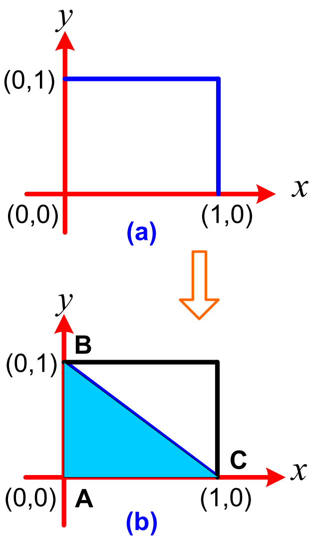


Figure 1.1

a. The value of k
It is given that $f_{xy}(x,y) = kxy$ is joint pdf, then

$$\int_{-\infty}^{-\infty}\int_{-\infty}^{-\infty} f(x,y)\ kxydxdy = 1$$

$$\begin{aligned}
\int_0^1\int_0^1 kxy\ dxdy &= k\int_0^1\left[\int_0^1 xydx\right]dy \\
&= k\int_0^1 [\frac{x^2}{2}y]_0^1\ dy \\
1 &= k\int_0^1 \frac{y}{2}\ dy \\
&= k[\frac{y^2}{4}]_0^1 = \frac{1}{4}k \\
k &= 4
\end{aligned}$$

$$f_{XY}(x,y) = 4xy \quad 0 < x < 2 \quad 0 < y < 2$$

b. Are X and Y independent ?

$$\begin{aligned}
f_X(x) &= k\int_0^1 xy\ dy \\
&= k[x\frac{y^2}{2}]_0^1 \\
&= k[x\frac{1}{2}] = 4[\frac{x}{2}] \\
&= 2x \quad 0 < x < 1
\end{aligned}$$

By symmetry

$$f_Y(y) = 2y \quad 0 < x < 1$$

$$\begin{aligned}
f_X(x)f_Y(y) &= 2x \times 2y \\
&= 4xy \\
f_{XY}(x,y) &= 4xy \\
f_{XY}(x,y) &= f_X(x)f_Y(y)
\end{aligned}$$

Hence X and Y are independent

c. $P(X+Y<1)$
The details of the limits are as shown in Figure 1.1 (b) By taking line BC. Considering y varies from 0 to 1 and x is a variable its lower limit is 0 and its upper limit is

$x_1 = 0,\ y_1 = 1,\ x_2 = 1,\ y_2 = 0$

$$\begin{aligned}
y - y_1 &= \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \\
y - 1 &= \frac{0-1}{1-0}(x-0) \\
y - 1 &= -x \\
x &= 1 - y
\end{aligned}$$

$$\begin{aligned}
\int_0^1\int_0^{1-y} kxy\ dxdy &= ky\int_0^1\left[\int_0^{1-y} xydx\right]dy \\
&= ky\int_0^1 [\frac{x^2}{2}]_0^{1-y}\ dy \\
&= 4y\int_0^1 \frac{1}{2}(1-y)^2\ dy \\
&= 2\int_0^1 y(1-2y+y^2)\ dy \\
&= 2\int_0^1 (y-2y^2+y^3)\ dy \\
&= 2\left[\frac{y^2}{2} - 2\frac{y^3}{3} + \frac{y^4}{4}]\right]_0^1 \\
&= \frac{1}{6}
\end{aligned}$$

2. The joint pdf $f_{XY}(x, y) = c$ a constant, when $0 < x < 3$ and $0 < y < 3$, and is 0 otherwise

[a.] **What is the value of of the constant c ?**

[b.] **What are the pdf for X and Y ?**

[c.] **What is $F_{XY}(x, y)$ when $0 < x < 3$ and $0 < y < 3$?**

[d.] **What are $F_{XY}(x, \infty)$ and $F_{xy}(\infty, y)$?**

[e.] **Are X and Y independent ?** [?]

Solution:

a.What is the value of of the constant c
It is given that $f_{XY}(x, y) = c$ is joint pdf, then

$$\begin{aligned}
\int_{-\infty}^{-\infty} f(x,y)\, dxdy &= 1 \\
\int_0^3 \int_0^3 c\, dxdy &= c\int_0^3 \left[\int_0^3 1dx\right] dy \\
&= c\int_0^3 [x]_0^3\, dy \\
1 &= c\int_0^3 3\, dy = 3c[y]_0^3 \\
1 &= 9c \\
c &= \frac{1}{9} \\
f_{XY}(x,y) &= \frac{1}{9}
\end{aligned}$$

b. What are the pdf for X and Y ?

$$\begin{aligned}
f_X(x) &= c\int_0^3 1\, dy \\
&= c[y]_0^3 = c \times 3 \\
&= \frac{1}{3} \quad 0 < x < 3
\end{aligned}$$

$$\begin{aligned}
f_Y(y) &= c\int_0^3 1\, dx \\
&= c[y]_0^3 = c \times 3 \\
&= \frac{1}{3} \quad 0 < y < 3
\end{aligned}$$

c.

$$\begin{aligned}
F_{XY}(x,y) &= c\int_0^x \int_0^y dudv \\
&= c\int_0^x \left[\int_0^y du\right] dv \\
&= c\int_0^x [u]_0^y\, dv \\
&= cy\int_0^x dv = cy\,[v]_0^x \\
&= \frac{1}{9}xy \quad 0 < x < 3, \quad 0 < y < 3,
\end{aligned}$$

d.

$$\begin{aligned}
F_X(x) &= F_{XY}(x,\infty) = c\int_0^x \int_0^3 dudv \\
&= c\int_0^x \left[\int_0^3 du\right] dv \\
&= c\int_0^x [y]_0^3\, dv \\
&= 3c\int_0^x dv = 3c\,[v]_0^x \\
&= 3\frac{1}{9}x \;= \frac{x}{3} \quad 0 < x < 3
\end{aligned}$$

$$\begin{aligned}
F_Y(y) &= F_{XY}(\infty,y) = c\int_0^3 \int_0^y dudv \\
&= c\int_0^3 \left[, \int_0^y du\right] dv \\
&= c\int_0^3 [y]_0^y\, dv \\
&= yc\int_0^3 dv = yc\,[v]_0^3 \\
&= 3\frac{1}{9}y \;= \frac{y}{3} \quad 0 < y < 3
\end{aligned}$$

e.
From the above equations it is observed that

$$\begin{aligned}
f_X(x)f_Y(y) &= \frac{1}{3} \times \frac{1}{3} = \frac{1}{9} \\
f_{XY}(x,y) &= \frac{1}{9} \\
f_{XY}(x,y) &= f_X(x)f_Y(y)
\end{aligned}$$

Therefore X and Y are independent. Similarly it is observed that

$$F_X(x)F_Y(y) = F_{XY}(x,y)$$

3. The joint pdf $f_{xy}(x,y) = c$ a constant, when $0 < x < 3$ and $0 < y < 4$, and is 0 otherwise

[a.] What is the value of of the constant c ?

[b.] What are the pdf for X and Y ?

[c.] What is $F_{xy}(x,y)$ when $0 < x < 3$ and $0 < y < 4$?

[d.] What are $F_{xy}(x,\infty)$ and $F_{xy}(\infty,y)$?

[e.] Are X and Y independent ? [?]

Solution:

a.
It is given that $f_{xy}(x,y) = c$ is joint pdf, then

$$\int_{-\infty}^{-\infty} f(x,y)\ dxdy = 1$$

$$\begin{aligned}
\int_0^4 \int_0^3 c\ dxdy &= c\int_0^4 \left[\int_0^3 1dx\right]\ dy \\
&= c\int_0^4 [x]_0^3\ dy \\
1 &= c\int_0^4 3\ dy = 3c[y]_0^4 \\
1 &= 12c \\
c &= \frac{1}{12}
\end{aligned}$$

b.

$$\begin{aligned}
f_X(x) &= c\int_0^4 1\ dy \\
&= c[y]_0^4 = c\times 4 \\
&= \frac{1}{3} \quad 0 < x < 3
\end{aligned}$$

$$\begin{aligned}
f_Y(y) &= c\int_0^3 1\ dx \\
&= c[y]_0^3 = c\times 3 \\
&= \frac{1}{4} \quad 0 < y < 4
\end{aligned}$$

c.

$$\begin{aligned}
F_{XY}(x,y) &= c\int_0^x \int_0^y dudv \\
&= c\int_0^x \left[\int_0^y du\right]\ dv \\
&= c\int_0^x [u]_0^y\ dv \\
&= cy\int_0^x dv = cy\,[v]_0^x \\
&= \frac{1}{12}xy \quad 0 < x < 3, \quad 0 < y < 4,
\end{aligned}$$

d.

$$\begin{aligned}
F_X(x) &= F_{XY}(x,\infty) = c\int_0^x \int_0^4 dudv \\
&= c\int_0^x \left[\int_0^4 du\right]\ dv \\
&= c\int_0^x [y]_0^4\ dv \\
&= 4c\int_0^x dv = 4c\,[v]_0^x \\
&= 4\frac{1}{12}x \ = \frac{x}{3} \quad 0 < x < 3
\end{aligned}$$

$$\begin{aligned}
F_Y(y) &= F_{XY}(\infty,y) = c\int_0^3 \int_0^y dudv \\
&= c\int_0^3 \left[,\int_0^y du\right]\ dv \\
&= c\int_0^3 [y]_0^y\ dv \\
&= yc\int_0^3 dv = yc\,[v]_0^3 \\
&= 3\frac{1}{12}y \ = \frac{y}{4} \quad 0 < y < 3
\end{aligned}$$

From the above equations it is observed that

$$f_X(x)f_Y(y) = f_{XY}(x,y)$$

Therefore X and Y are independent. Similarly it is observed that

$$F_X(x)F_Y(y) = F_{XY}(x,y)$$

4. The joint pdf $f_{xy}(x,y) = c$ a constant, when $0 < x < 2$ and $0 < y < 3$, and is 0 otherwise

[a.] What is the value of of the constant c ?

[b.] What are the pdf for X and Y ?

[c.] What is $F_{xy}(x,y)$ when $0 < x < 2$ and $0 < y < 3$?

[d.] What are $F_{xy}(x,\infty)$ and $F_{xy}(\infty,y)$?

[e.] Are X and Y independent ? [?]

Solution:

a.

It is given that $f_{xy}(x,y) = c$ is joint pdf, then

$$\int_{-\infty}^{-\infty} f(x,y)\ dxdy = 1$$

$$\begin{aligned}
\int_0^3 \int_0^2 c\ dxdy &= c\int_0^3 \left[\int_0^2 1dx\right]\ dy \\
&= c\int_0^3 [x]_0^2\ dy \\
1 &= c\int_0^3 2\ dy = 2c[y]_0^3 \\
1 &= 6c \\
c &= \frac{1}{6}
\end{aligned}$$

b.

$$\begin{aligned}
f_X(x) &= c\int_0^3 1\ dy \\
&= c[y]_0^3 = c \times 3 \\
&= \frac{1}{2} \quad 0 < x < 2
\end{aligned}$$

$$\begin{aligned}
f_Y(y) &= c\int_0^2 1\ dx \\
&= c[y]_0^2 = c \times 2 \\
&= \frac{1}{3} \quad 0 < y < 3
\end{aligned}$$

c.

$$\begin{aligned}
F_{XY}(x,y) &= c\int_0^x \int_0^y dudv \\
&= c\int_0^x \left[\int_0^y du\right]\ dv \\
&= c\int_0^x [u]_0^y\ dv \\
&= cy\int_0^x dv = cy\,[v]_0^x \\
&= \frac{1}{6}xy \quad 0 < x < 2, \quad 0 < y < 3,
\end{aligned}$$

d.

$$\begin{aligned}
F_X(x) &= F_{XY}(x,\infty) = c\int_0^x \int_0^3 dudv \\
&= c\int_0^x \left[\int_0^3 du\right]\ dv \\
&= c\int_0^x [y]_0^3\ dv \\
&= 4c\int_0^x dv = 3c\,[v]_0^x \\
&= 4\frac{1}{6}x \ = \frac{x}{2} \quad 0 < x < 2
\end{aligned}$$

$$\begin{aligned}
F_Y(y) &= F_{XY}(\infty,y) = c\int_0^2 \int_0^y dudv \\
&= c\int_0^2 \left[\int_0^y du\right]\ dv \\
&= c\int_0^2 [y]_0^y\ dv \\
&= yc\int_0^2 dv = yc\,[v]_0^2 \\
&= 3\frac{1}{6}y \ = \frac{y}{3} \quad 0 < y < 3
\end{aligned}$$

From the above equations it is observed that

$$f_X(x)f_Y(y) = f_{XY}(x,y)$$

Therefore X and Y are independent. Similarly it is observed that

$$F_X(x)F_Y(y) = F_{XY}(x,y)$$

5. A bivariate pdf for the discrete random variables. X and Y is

$$0.2\delta(x)\delta(y) + 0.3\delta(x-1)\delta(y) + 0.3\delta(x)\delta(y-1) + c\delta(x-1)\delta(y-1)$$

[a.] **What is the value of of the constant c ?**

[b.] **What are the pdf for X and Y ?**

[c.] **What is $F_{XY}(x, y)$ when $0 < x < 1$ and $0 < y < 1$?**

[d.] **What are $F_{XY}(x, \infty)$ and $F_{XY}(\infty, y)$?**

[e.] **Are X and Y independent ?** [?]

Solution:

$$f_{XY}(x, y) = 0.2\delta(x)\delta(y) + 0.3\delta(x-1)\delta(y) + 0.3\delta(x)\delta(y-1) + c\delta(x-1)\delta(y-1)$$

a.
It is given that the given function is bivariate pdf then,

$$\begin{aligned} 1 &= 0.2 + 0.3 + 0.3 + c \\ c &= 1 - 0.8 \\ c &= 0.2 \end{aligned}$$

Hence given function is

$$0.2\delta(x)\delta(y) + 0.3\delta(x-1)\delta(y) + 0.3\delta(x)\delta(y-1) + 0.2\delta(x-1)\delta(y-1)$$

b.

$$\begin{aligned} f_X(x) &= 0.2\delta(x) + 0.3\delta(x-1) + 0.3\delta(x) + 0.2\delta(x-1) \\ &= 0.5\delta(x) + 0.5\delta(x-1) \end{aligned}$$

$$\begin{aligned} f_Y(y) &= 0.2\delta(y) + 0.3\delta(y) + 0.3\delta(y-1) + 0.2\delta(y-1) \\ &= 0.5\delta(y) + 0.5\delta(y-1) \end{aligned}$$

c.

$$F_{XY}(x, y) = 0.2 \quad 0 < x < 1 \; and \; 0 < y < 1$$

d.

$$\begin{aligned} f(x, y) &= 0.2\delta(x)\delta(y) + 0.3\delta(x-1)\delta(y) + 0.3\delta(x)\delta(y-1) + 0.2\delta(x-1)\delta(y-1) \\ F_X(x) &= 0.5u(x) + 0.5u(x-1) \\ F_Y(y) &= 0.5u(y) + 0.5u(y-1) \end{aligned}$$

e.

$$\begin{aligned} f_X(x)f_Y(y) &= [0.5\delta(x) + 0.5\delta(x-1)][0.5\delta(y) + 0.5\delta(y-1)] \\ &= 0.25\delta(x)\delta(y) + 0.25\delta(x-1)\delta(y) + 0.25\delta(x)\delta(y-1) + 0.25\delta(x-1)\delta(y-1) \end{aligned}$$

From the above equations it is observed that

$$f_X(x)f_Y(y) \neq f_{XY}(x, y)$$

Therefore X and Y are not independent.

6. A bivariate pdf for the discrete random variables. X and Y is

$$0.3\delta(x)\delta(y) + 0.2\delta(x-1)\delta(y) + 0.3\delta(x)\delta(y-1) + c\delta(x-1)\delta(y-1)$$

[a.] **What is the value of of the constant c ?**

[b.] **What are the pdf for X and Y ?**

[c.] **What is $F_{XY}(x, y)$ when $0 < x < 1$ and $0 < y < 1$?**

[d.] **What are $F_{XY}(x, \infty)$ and $F_{XY}(\infty, y)$?**

[e.] **Are X and Y independent ?** [?]

Solution:

$$f_{XY}(x, y) = 0.3\delta(x)\delta(y) + 0.2\delta(x-1)\delta(y) + 0.3\delta(x)\delta(y-1) + c\delta(x-1)\delta(y-1)$$

a.
It is given that the given function is bivariate pdf then,

$$\begin{aligned} 1 &= 0.3 + 0.2 + 0.3 + c \\ c &= 1 - 0.8 \\ c &= 0.2 \end{aligned}$$

Hence given function is

$$f_{XY}(x, y) = 0.3\delta(x)\delta(y) + 0.2\delta(x-1)\delta(y) + 0.3\delta(x)\delta(y-1) + 0.2\delta(x-1)\delta(y-1)$$

b.

$$\begin{aligned} f_X(x) &= 0.3\delta(x) + 0.2\delta(x-1) + 0.3\delta(x) + 0.2\delta(x-1) \\ &= 0.6\delta(x) + 0.4\delta(x-1) \end{aligned}$$

$$\begin{aligned} f_Y(y) &= 0.3\delta(y) + 0.2\delta(y) + 0.3\delta(y-1) + 0.2\delta(y-1) \\ &= 0.5\delta(y) + 0.5\delta(y-1) \end{aligned}$$

c.

$$F_{XY}(x, y) = 0.3 \quad 0 < x < 1 \ and \ \ 0 < y < 1$$

d.

$$\begin{aligned} f(x, y) &= 0.3\delta(x)\delta(y) + 0.2\delta(x-1)\delta(y) + 0.3\delta(x)\delta(y-1) + 0.2\delta(x-1)\delta(y-1) \\ F_X(x) &= 0.6u(x) + 0.4u(x-1) \\ F_Y(y) &= 0.5u(y) + 0.5u(y-1) \end{aligned}$$

e.

$$\begin{aligned} f_X(x)f_Y(y) &= [0.6\delta(x) + 0.4\delta(x-1)][0.5\delta(y) + 0.5\delta(y-1)] \\ &= 0.3\delta(x)\delta(y) + 0.2\delta(x-1)\delta(y) + 0.3\delta(x)\delta(y-1) + 0.2\delta(x-1)\delta(y-1) \end{aligned}$$

From the above equations it is observed that

$$f_X(x)f_Y(y) \neq f_{XY}(x, y)$$

Therefore X and Y are not independent.

7. A bivariate pdf for the discrete random variables. X and Y is

$$0.2\delta(x)\delta(y) + 0.3\delta(x-1)\delta(y) + 0.2\delta(x)\delta(y-1) + c\delta(x-1)\delta(y-1)$$

[a.] **What is the value of of the constant c ?**

[b.] **What are the pdf for X and Y ?**

[c.] **What is $F_{XY}(x, y)$ when $0 < x < 1$ and $0 < y < 1$?**

[d.] **What are $F_{XY}(x, \infty)$ and $F_{XY}(\infty, y)$?**

[e.] **Are X and Y independent ?** [?]

Solution:

$$f_{XY}(x,y) = 0.2\delta(x)\delta(y) + 0.3\delta(x-1)\delta(y) + 0.2\delta(x)\delta(y-1) + c\delta(x-1)\delta(y-1)$$

a.
It is given that the given function is bivariate pdf then,

$$\begin{aligned} 1 &= 0.2 + 0.3 + 0.2 + c \\ c &= 1 - 0.7 = 0.3 \end{aligned}$$

Hence given function is

$$f_{XY}(x,y) = 0.2\delta(x)\delta(y) + 0.3\delta(x-1)\delta(y) + 0.2\delta(x)\delta(y-1) + c\delta(x-1)\delta(y-1)$$

b.

$$\begin{aligned} f_X(x) &= 0.2\delta(x) + 0.3\delta(x-1) + 0.2\delta(x) + 0.2\delta(x-1) \\ &= 0.4\delta(x) + 0.6\delta(x-1) \end{aligned}$$

$$\begin{aligned} f_Y(y) &= 0.2\delta(y) + 0.3\delta(y) + 0.2\delta(y-1) + 0.3\delta(y-1) \\ &= 0.5\delta(y) + 0.5\delta(y-1) \end{aligned}$$

c.

$$F_{XY}(x,y) = 0.2 \quad 0 < x < 1 \ and \ 0 < y < 1$$

d.

$$\begin{aligned} f(x,y) &= 0.2\delta(x)\delta(y) + 0.3\delta(x-1)\delta(y) + 0.2\delta(x)\delta(y-1) + c\delta(x-1)\delta(y-1) \\ F_X(x) &= 0.4u(x) + 0.6u(x-1) \\ F_Y(y) &= 0.5u(y) + 0.5u(y-1) \end{aligned}$$

e.

$$\begin{aligned} f_X(x)f_Y(y) &= [0.4\delta(x) + 0.6\delta(x-1)][0.5\delta(y) + 0.5\delta(y-1)] \\ &= 0.2\delta(x)\delta(y) + 0.3\delta(x-1)\delta(y) + 0.2\delta(x)\delta(y-1) + 0.3\delta(x-1)\delta(y-1) \end{aligned}$$

From the above equations it is observed that

$$f_X(x)f_Y(y) \neq f_{XY}(x,y)$$

Therefore X and Y are not independent.

Example 3.5. Given A bivariate pdf for the discrete random variables. X and Y is

$$f_{XY}(xy) = \frac{1}{1.4283\pi} exp\left[-\frac{(x^2-1.4xy+y^2)}{1.02}\right] \qquad -\infty < x,\ y < \infty$$

[a.] **What are the pdf for X and Y ?**

[b.] $F_{XY}(\infty,\infty) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1$

[c.] **Are X and Y independent ?** [?]

Solution:

a) The pdf for X and Y
$a = 1, b = 1.4, c = 1$

$$\begin{aligned} x^2 - 1.4xy + y^2 &= y^2 - 2\times 0.7xy + 0.49x^2 + 0.51x^2 \\ &= (y-0.7x)^2 + 0.51x^2 \end{aligned}$$

$$f_{XY}(xy) = \frac{1}{1.4283\pi} exp\left[-\frac{(y-0.7x)^2+0.51x^2)}{1.02}\right] \qquad -\infty < x,\ y < \infty$$

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(xy)dy \\ &= \frac{1}{1.4283\pi} e^{-0.5x^2} \int_{-\infty}^{\infty} exp\left[-\frac{(y-0.7x)^2}{1.02}\right] dy \end{aligned}$$

$$\frac{u}{\sqrt{2}} = \frac{y-0.7x}{\sqrt{1.02}}$$
$$\sqrt{\frac{1.02}{2}}u = y - 0.7x$$
$$\sqrt{\frac{1.02}{2}}du = dy$$

Also

$$\begin{aligned} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}dz &= 1 \\ \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}dz &= \sqrt{2\pi} \end{aligned}$$

$$\begin{aligned} f_X(x) &= \frac{1}{1.4283\pi} e^{-0.5x^2}\sqrt{\frac{1.02}{2}}\int_{-\infty}^{\infty} exp\left[-\frac{u^2}{2}\right] du \\ &= \frac{1}{1.4283\pi} e^{-0.5x^2}\sqrt{\frac{1.02}{2}}\sqrt{2\pi} \\ &= \frac{1}{\sqrt{2\pi}} e^{-0.5x^2} \\ &= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \end{aligned}$$

$$\begin{aligned} x^2 - 1.4xy + y^2 &= x^2 - 2\times 0.7xy + 0.49y^2 + 0.51y^2 \\ &= (x-0.7y)^2 + 0.51y^2 \end{aligned}$$

a) The pdf for Y

$$f_{XY}(xy) = \frac{1}{1.4283\pi} exp\left[-\frac{((x-0.7y)^2+0.51y^2)}{1.02}\right] \quad -\infty < x,\ y < \infty$$

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(xy)dy \\ &= \frac{1}{1.4283\pi} e^{-0.5y^2} \int_{-\infty}^{\infty} exp\left[-\frac{(x-0.7y)^2}{1.02}\right] dx \end{aligned}$$

$$\begin{aligned} \frac{u}{\sqrt{2}} &= \frac{x-0.7y}{\sqrt{1.02}} \\ \sqrt{\frac{1.02}{2}}u &= x-0.7y \\ \sqrt{\frac{1.02}{2}}du &= dx \end{aligned}$$

Also

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz &= 1 \\ \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz &= \sqrt{2\pi} \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \frac{1}{1.4283\pi} e^{-0.5y^2} \sqrt{\frac{1.02}{2}} \int_{-\infty}^{\infty} exp\left[-\frac{u^2}{2}\right] du \\ &= \frac{1}{1.4283\pi} e^{-0.5y^2} \sqrt{\frac{1.02}{2}} \sqrt{2\pi} \\ &= \frac{1}{\sqrt{2\pi}} e^{-0.5y^2} \\ &= \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \end{aligned}$$

b.
$F_{XY}(\infty,\infty) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1$

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(xy)dxdy &= \\ &= \int_{-\infty}^{\infty} f_X(x)\left[\int_{-\infty}^{\infty} f_Y(y)dy\right]dx \\ &= \int_{-\infty}^{\infty} f_X(x)\left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}dy\right]dx \\ &= \int_{-\infty}^{\infty} f_X(x)dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}dx \\ &= 1 \end{aligned}$$

c. Are X and Y independent

$$f_{XY}(xy) = \frac{1}{1.4283\pi} exp\left[-\frac{(x^2-1.4xy+y^2)}{1.02}\right] \quad -\infty < x,\ y < \infty$$

$$\begin{aligned} f_X(x)f_Y(y) &= \left[\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right]\left[\frac{1}{\sqrt{2\pi}}e^{-y^2/2}\right] \\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2+y^2}{2}} \end{aligned}$$

$$f_{XY}(xy) \neq f_X(x)f_Y(y)$$

Bivariate random variables X and Y are not independent

8. Given A bivariate pdf for the discrete random variables. X and Y is

$$f_{XY}(xy) = \frac{1}{1.4283\pi} exp\left[-\frac{(x^2+1.4xy+y^2)}{1.02}\right] \quad -\infty < x,\ y < \infty$$

[a.] **What are the pdf for X and Y ?**

[b.] $F_{XY}(\infty,\infty) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1$

[c.] **Are X and Y independent ?** [?]

Solution: $a = 1, b = 1.4, c = 1$

a.

$$\begin{aligned} x^2+1.4xy+y^2 &= y^2 + 2\times 0.7xy + (0.7x)^2 + 0.51x^2 \\ &= (y+0.7x)^2 + 0.51x^2 \end{aligned}$$

$$f_{XY}(xy) = \frac{1}{1.4283\pi} exp\left[-\frac{(y+0.7x)^2+0.51x^2)}{1.02}\right] \quad -\infty < x,\ y < \infty$$

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(xy)dy \\ &= \frac{1}{1.4283\pi}e^{-0.5x^2}\int_{-\infty}^{\infty} exp\left[-\frac{(y+0.7x)^2}{1.02}\right]dy \end{aligned}$$

$$\frac{u}{\sqrt{2}} = \frac{y+0.7x}{\sqrt{1.02}}$$

$$\sqrt{\frac{1.02}{2}}u = y+0.7x$$

$$\sqrt{\frac{1.02}{2}}du = dy$$

Also

$$\begin{aligned} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}dz &= 1 \\ \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}dz &= \sqrt{2\pi} \end{aligned}$$

$$\begin{aligned} f_X(x) &= \frac{1}{1.4283\pi} e^{-0.5x^2} \sqrt{\frac{1.02}{2}} \int_{-\infty}^{\infty} exp\left[-\frac{u^2}{2}\right] du \\ &= \frac{1}{1.4283\pi} e^{-0.5x^2} \sqrt{\frac{1.02}{2}} \sqrt{2\pi} \\ &= \frac{1}{\sqrt{2\pi}} e^{-0.5x^2} \\ &= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \end{aligned}$$

$$\begin{aligned} x^2 + 1.4xy + y^2 &= x^2 + 2 \times 0.7xy + (0.7y)^2 + 0.51y^2 \\ &= (x + 0.7y)^2 + 0.51y^2 \end{aligned}$$

$$f_{XY}(xy) = \frac{1}{1.4283\pi} exp\left[-\frac{(x + 0.7y)^2 + 0.51y^2)}{1.02}\right] \quad -\infty < x,\ y < \infty$$

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(xy) dy \\ &= \frac{1}{1.4283\pi} e^{-0.5y^2} \int_{-\infty}^{\infty} exp\left[-\frac{(x + 0.7y)^2}{1.02}\right] dx \end{aligned}$$

$$\begin{aligned} \frac{u}{\sqrt{2}} &= \frac{x + 0.7y}{\sqrt{1.02}} \\ \sqrt{\frac{1.02}{2}} u &= x + 0.7y \\ \sqrt{\frac{1.02}{2}} du &= dx \end{aligned}$$

Also

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz &= 1 \\ \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz &= \sqrt{2\pi} \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \frac{1}{1.4283\pi} e^{-0.5y^2} \sqrt{\frac{1.02}{2}} \int_{-\infty}^{\infty} exp\left[-\frac{u^2}{2}\right] du \\ &= \frac{1}{1.4283\pi} e^{-0.5y^2} \sqrt{\frac{1.02}{2}} \sqrt{2\pi} \\ &= \frac{1}{\sqrt{2\pi}} e^{-0.5y^2} \\ &= \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \end{aligned}$$

b.

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(xy)dxdy &= \\
&= \int_{-\infty}^{\infty} f_X(x)\left[\int_{-\infty}^{\infty} f_Y(y)dy\right]dx \\
&= \int_{-\infty}^{\infty} f_X(x)\left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy\right]dx \\
&= \int_{-\infty}^{\infty} f_X(x)dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \\
&= 1
\end{aligned}$$

c.

$$f_{XY}(xy) = \frac{1}{1.4283\pi}exp\left[-\frac{(x^2+1.4xy+y^2)}{1.02}\right] \qquad -\infty < x,\ y < \infty$$

$$\begin{aligned}
f_X(x)f_Y(y) &= \left[\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right]\left[\frac{1}{\sqrt{2\pi}}e^{-y^2/2}\right] \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2+y^2}{2}}
\end{aligned}$$

$$f_{XY}(xy) \neq f_X(x)f_Y(y)$$

Bivariate random variables X and Y are not independent

9. Given A bivariate pdf for the discrete random variables. X and Y is

$$f_{XY}(xy) = \frac{1}{1.9079\pi}exp\left[-\frac{(x^2-0.6xy+y^2)}{1.82}\right] \qquad -\infty < x,\ y < \infty$$

[a.] **What are the pdf for X and Y ?**

[b.] $F_{XY}(\infty,\infty)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1$

[c.] **Are X and Y independent ?** [?]

Solution: $a = 1, b = 1.4, c = 1$

a.

$$\begin{aligned}
x^2 - 0.6xy + y^2 &= y^2 - 2\times 0.3xy + (0.3x)^2 + 0.91x^2 \\
&= (y-0.3x)^2 + 0.91x^2
\end{aligned}$$

$$f_{XY}(xy) = \frac{1}{1.9079\pi}exp\left[-\frac{(y-0.3x)^2+0.91x^2)}{1.82}\right] \qquad -\infty < x,\ y < \infty$$

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(xy)dy \\
&= \frac{1}{1.9079\pi}e^{-0.5x^2}\int_{-\infty}^{\infty} exp\left[-\frac{(y-0.3x)^2}{1.82}\right]dy
\end{aligned}$$

$$\frac{u}{\sqrt{2}} = \frac{y - 0.3x}{\sqrt{1.82}}$$
$$\sqrt{\frac{1.82}{2}}u = y - 0.3x$$
$$\sqrt{\frac{1.82}{2}}du = dy$$

Also

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}dz &= 1\\ \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}dz &= \sqrt{2\pi}\end{aligned}$$

$$\begin{aligned}f_X(x) &= \frac{1}{1.9079\pi}e^{-0.5x^2}\sqrt{\frac{1.82}{2}}\int_{-\infty}^{\infty} exp\left[-\frac{u^2}{2}\right]du\\ &= \frac{1}{1.9079\pi}e^{-0.5x^2}\sqrt{\frac{1.82}{2}}\sqrt{2\pi}\\ &= \frac{1}{\sqrt{2\pi}}e^{-0.5x^2}\\ &= \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\end{aligned}$$

$$\begin{aligned}x^2 - 0.6xy + y^2 &= x^2 - 2\times 0.3xy - (0.3y)^2 + 0.91y^2\\ &= (x - 0.3y)^2 + 0.91y^2\end{aligned}$$

$$f_{XY}(xy) = \frac{1}{1.9079\pi}exp\left[-\frac{(x-0.3y)^2 + 0.91y^2)}{1.82}\right] \quad -\infty < x,\ y < \infty$$

$$\begin{aligned}f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(xy)dy\\ &= \frac{1}{1.9079\pi}e^{-0.5y^2}\int_{-\infty}^{\infty} exp\left[-\frac{(x-0.3y)^2}{1.92}\right]dx\end{aligned}$$

$$\frac{u}{\sqrt{2}} = \frac{x - 0.3y}{\sqrt{1.92}}$$
$$\sqrt{\frac{1.02}{2}}u = x - 0.3y$$
$$\sqrt{\frac{1.92}{2}}du = dx$$

Also

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}dz &= 1\\ \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}dz &= \sqrt{2\pi}\end{aligned}$$

$$
\begin{aligned}
f_Y(y) &= \frac{1}{1.9079\pi}e^{-0.5y^2}\sqrt{\frac{1.92}{2}}\int_{-\infty}^{\infty} exp\left[-\frac{u^2}{2}\right]du \\
&= \frac{1}{1.9079\pi}e^{-0.5y^2}\sqrt{\frac{1.92}{2}}\sqrt{2\pi} \\
&= \frac{1}{\sqrt{2\pi}}e^{-0.5y^2} \\
&= \frac{1}{\sqrt{2\pi}}e^{-y^2/2}
\end{aligned}
$$

b.

$$
\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(xy)dxdy &= \\
&= \int_{-\infty}^{\infty} f_X(x)\left[\int_{-\infty}^{\infty} f_Y(y)dy\right]dx \\
&= \int_{-\infty}^{\infty} f_X(x)\left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy\right]dx \\
&= \int_{-\infty}^{\infty} f_X(x)dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \\
&= 1
\end{aligned}
$$

c.

$$
f_{XY}(xy) = \frac{1}{1.9079\pi}exp\left[-\frac{(x^2-0.3xy+(0.3y)^2)}{1.92}\right] \qquad -\infty < x,\ y < \infty
$$

$$
\begin{aligned}
f_X(x)f_Y(y) &= \left[\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right]\left[\frac{1}{\sqrt{2\pi}}e^{-y^2/2}\right] \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2+y^2}{2}}
\end{aligned}
$$

$$
f_{XY}(xy) \neq f_X(x)f_Y(y)
$$

Bivariate random variables X and Y are not independent

10. Given A bivariate pdf for the discrete random variables. X and Y is

$$
f_{XY}(xy) = \frac{1}{1.7321\pi}exp\left[-\frac{(x^2+1.0xy+y^2)}{1.5}\right] \qquad -\infty < x,\ y < \infty
$$

[a.] **What are the pdf for X and Y ?**

[b.] $F_{XY}(\infty,\infty)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1$

[c.] **Are X and Y independent ?** [?]

Solution:

a.

$$
\begin{aligned}
x^2+1.0xy+y^2 &= y^2+2\times 0.5xy+(0.5x)^2+0.75x^2 \\
&= (y+0.5x)^2+0.75x^2
\end{aligned}
$$

$$f_{XY}(xy) = \frac{1}{1.7321\pi} exp\left[-\frac{(y+0.5x)^2+0.75x^2)}{1.50}\right] \quad -\infty < x,\ y < \infty$$

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(xy)dy \\ &= \frac{1}{1.7321\pi} e^{-0.5x^2} \int_{-\infty}^{\infty} exp\left[-\frac{(y+0.5x)^2}{1.50}\right] dy \end{aligned}$$

$$\frac{u}{\sqrt{2}} = \frac{y+0.5x}{\sqrt{1.50}}$$
$$\sqrt{\frac{1.50}{2}} u = y+0.5x$$
$$\sqrt{\frac{1.50}{2}} du = dy$$

Also

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz &= 1 \\ \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz &= \sqrt{2\pi} \end{aligned}$$

$$\begin{aligned} f_X(x) &= \frac{1}{1.7321\pi} e^{-0.5x^2} \sqrt{\frac{1.50}{2}} \int_{-\infty}^{\infty} exp\left[-\frac{u^2}{2}\right] du \\ &= \frac{1}{1.7321\pi} e^{-0.5x^2} \sqrt{\frac{1.50}{2}} \sqrt{2\pi} \\ &= \frac{1}{\sqrt{2\pi}} e^{-0.5x^2} \\ &= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \end{aligned}$$

$$\begin{aligned} x^2+1.0xy+y^2 &= x^2+2\times 0.5xy+(0.5y)^2+0.75y^2 \\ &= (x+0.5y)^2+0.75y^2 \end{aligned}$$

$$f_{XY}(xy) = \frac{1}{1.7321\pi} exp\left[-\frac{(x+0.5y)^2+0.75y^2)}{1.50}\right] \quad -\infty < x,\ y < \infty$$

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(xy)dy \\ &= \frac{1}{1.7321\pi} e^{-0.5y^2} \int_{-\infty}^{\infty} exp\left[-\frac{(x+0.5y)^2}{1.50}\right] dy \end{aligned}$$

$$\frac{u}{\sqrt{2}} = \frac{x+0.5y}{\sqrt{1.50}}$$
$$\sqrt{\frac{1.50}{2}} u = x+0.5y$$
$$\sqrt{\frac{1.50}{2}} du = dx$$

Also

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}dz &= 1\\ \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}dz &= \sqrt{2\pi}\end{aligned}$$

$$\begin{aligned}f_X(x) &= \frac{1}{1.7321\pi}e^{-0.5x^2}\sqrt{\frac{1.50}{2}}\int_{-\infty}^{\infty} exp\left[-\frac{u^2}{2}\right]du\\ &= \frac{1}{1.7321\pi}e^{-0.5x^2}\sqrt{\frac{1.50}{2}}\sqrt{2\pi}\\ &= \frac{1}{\sqrt{2\pi}}e^{-0.5x^2}\\ &= \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\end{aligned}$$

b.

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(xy)dxdy &=\\ &= \int_{-\infty}^{\infty} f_X(x)\left[\int_{-\infty}^{\infty} f_Y(y)dy\right]dx\\ &= \int_{-\infty}^{\infty} f_X(x)\left[\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy\right]dx\\ &= \int_{-\infty}^{\infty} f_X(x)dx\\ &= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx\\ &= 1\end{aligned}$$

c.

$$f_{XY}(xy) = \frac{1}{1.7321\pi}exp\left[-\frac{(x^2+0.5xy+(0.5y)^2)}{1.50}\right] \quad -\infty < x,\ y < \infty$$

$$\begin{aligned}f_X(x)f_Y(y) &= \left[\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right]\left[\frac{1}{\sqrt{2\pi}}e^{-y^2/2}\right]\\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2+y^2}{2}}\end{aligned}$$

$$f_{XY}(xy) \neq f_X(x)f_Y(y)$$

Bivariate random variables X and Y are not independent

11 As shown in Figure is a region in the x, y plane where the bivariate pdf $f_{XY}(xy) = c$. Elsewhere the pdf is 0.

[a.] **What value must c have?**

[b.] **Evaluate $F_{XY}(1,1)$**

[c.] **Find the pdfs $f_X(x)$ and $f_Y(y)$.**

[d.] **Are X and Y independent ?** [?]

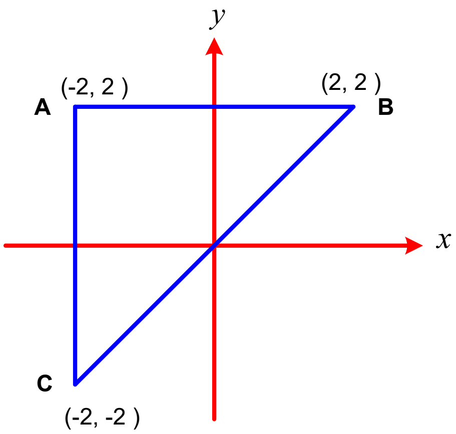


Figure 1.2

Integration Limits
By taking line CB. Considering x varies from -2 to 2 and y is a variable its upper limit is 2 and its lower lower limit is

$x_1 = -2,\ y_1 = -2,\ x_2 = 2,\ y_2 = 2$

$$\begin{aligned} y - y_1 &= \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \\ y - (-2) &= \frac{2 - (-2)}{2 - (-2)}(x - (-2)) \\ y + 2 &= x + 2 \\ y &= x \end{aligned}$$

Solution:

a.

$$\begin{aligned} F_{XY}(2,2) &= \int_{-2}^{2}\int_{x}^{2} c\ dydx \\ 1 &= c\int_{-2}^{2}\left[\int_{x}^{2} dy\right] dx \\ 1 &= c\int_{-2}^{2}[y]_x^2\,dx = c\int_{-2}^{2}[2-x]dx = c\left[2x - \frac{x^2}{2}\right]_{-2}^{2} \\ &= \frac{c}{2}\left[4x - x^2\right]_{-2}^{2} = \frac{c}{2}\left[[4\times 2 - (2)^2] - [4\times(-2) - (-2)^2]\right] \\ &= \frac{c}{2}[[8-4] - [-8-4]] = \frac{c}{2}[4+12] \\ 1 &= 8c \\ c &= \frac{1}{8} \end{aligned}$$

b.

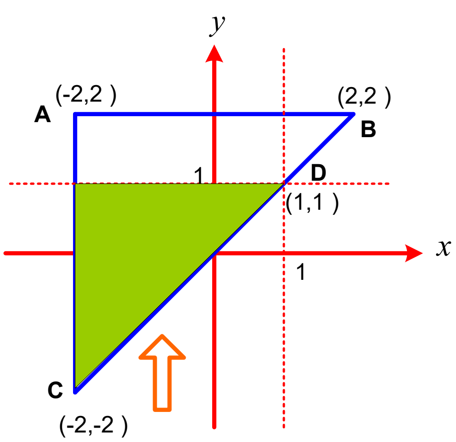


Figure 1.3

Integration Limits
By taking line CD. Considering x varies from -2 to 2 and y is a variable its upper limit is 1 and its lower lower limit is

$x_1 = -2,\ y_1 = -2,\ x_2 = 1,\ y_2 = 1$

$$\begin{aligned} y - y_1 &= \frac{1 - y_1}{1 - x_1}(x - x_1) \\ y - (-2) &= \frac{1 - (-2)}{1 - (-2)}(x - (-2)) \\ y + 2 &= x + 2 \\ y &= x \end{aligned}$$

$$
\begin{aligned}
F_{XY}(1,1) &= \int_{-2}^{1}\int_{x}^{1} c\ dydx \\
&= c\int_{-2}^{1}\left[\int_{x}^{1} dy\right] dx \\
&= c\int_{-2}^{1} [y]_x^1\, dx = c\int_{-2}^{1}[1-x]dx = c\int_{-2}^{1}\left[x-\frac{x^2}{2}\right]_{-2}^{1} dx \\
&= \frac{c}{2}\left[2x-x^2\right]_{-2}^{1} = \frac{c}{2}\left[[2\times 1-(1)^2]-[2\times(-2)-(-2)^2]\right] \\
&= \frac{c}{2}\left[[2-1]-[-4-4]\right] = \frac{c}{2}[1+8] \\
&= 9c \\
&= \frac{9}{16}
\end{aligned}
$$

c.

$$
\begin{aligned}
f_X(x) &= c\int_{x}^{2} dy \\
&= c\,[y]_x^2 = c\,[2-x] \\
&= \frac{1}{8}\,[2-x]
\end{aligned}
\qquad
\begin{aligned}
f_Y(y) &= c\int_{-2}^{y} dx \\
&= c\,[x]_{-2}^{y} = c\,[y+2] \\
&= \frac{1}{8}\,[[y+2]
\end{aligned}
$$

$$
\begin{aligned}
f_{XY}(x,y) &= \frac{1}{8} \\
f_X(x)f_Y(y) &= \frac{1}{8}\,[2-x]\,\frac{1}{8}\,[y+2] \\
f_{XY}(x,y) &\neq f_X(x)f_Y(y)
\end{aligned}
$$

12 As shown in Figure is a region in the x, y plane where the bivariate pdf $f_{XY}(xy) = c$. Elsewhere the pdf is 0.

[a.] **What value must c have?**

[b.] **Evaluate** $F_{XY}(1,1)$

[c.] **Find the pdfs** $f_X(x)$ **and** $f_Y(y)$.

[d.] **Are** X **and** Y **independent ?** [?]

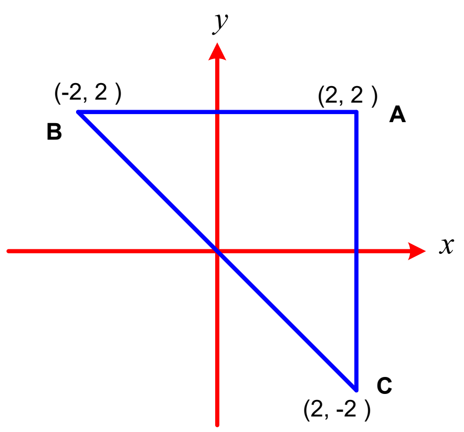


Figure 1.4

Integration Limits

By taking line BC. Considering x varies from -2 to 2 and y is a variable its upper limit is 2 and its lower lower limit is

$$x_1 = -2,\ y_1 = 2,\ x_2 = 2,\ y_2 = -2$$

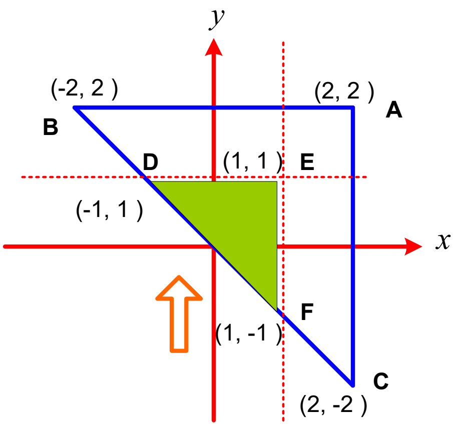
$$
\begin{aligned}
y - y_1 &= \frac{y_2-y_1}{x_2-x_1}(x-x_1) \\
y - 2 &= \frac{-2-2}{2+2}(x-(-2)) \\
y - 2 &= -x-2 \\
y &= -x
\end{aligned}
$$

Solution:

a.

$$\begin{aligned}
F_{XY}(2,2) &= \int_{-2}^{2}\int_{-x}^{2} c\; dydx \\
1 &= c\int_{-2}^{2}\left[\int_{-x}^{2} dy\right] dx \\
1 &= c\int_{-2}^{2} [y]_{-x}^{2}\, dx = c\int_{-2}^{2}[2+x]dx = c\left[2x - \frac{x^2}{2}\right]_{-2}^{2} \\
&= \frac{c}{2}\left[4x + x^2\right]_{-2}^{2} = \frac{c}{2}\left[[4\times 2-(2)^2] - [4\times(-2)-(-2)^2]\right] \\
&= \frac{c}{2}[[8-4]-[-8-4]] = \frac{c}{2}[4+12] \\
1 &= 8c \\
c &= \frac{1}{8}
\end{aligned}$$

b.



Integration Limits

By taking line DF. Considering x varies from -1 to 1 and y is a variable its upper limit is 1 and its lower lower limit is

$x_1 = -1,\ y_1 = 1,\ x_2 = 1,\ y_2 = -1$

$$\begin{aligned}
y - y_1 &= \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \\
y - 1 &= \frac{-1-1}{1+1}(x-(-1)) \\
y - 1 &= -x - 1 \\
y &= -x
\end{aligned}$$

$$\begin{aligned}
F_{XY}(1,1) &= \int_{-1}^{1}\int_{-x}^{1} c\; dydx \\
&= c\int_{-1}^{1}\left[\int_{-x}^{1} dy\right] dx \\
&= c\int_{-1}^{1} [y]_{-x}^{1}\, dx = c\int_{-1}^{1}[1+x]dx = c\left[x + \frac{x^2}{2}\right]_{-1}^{1} dx \\
&= \frac{c}{2}\left[2x + x^2\right]_{-1}^{1} = \frac{c}{2}\left[[2\times 1+(1)^1] - [2\times(-1)+(-1)^2]\right] \\
&= \frac{c}{2}[[2+1]-[-2+1]] = \frac{c}{2}[3+1] \\
&= 2c = 2\frac{1}{8} \\
&= \frac{1}{4}
\end{aligned}$$

c.

$$\begin{aligned}
f_X(x) &= c\int_{-x}^{2} dy \\
&= c\,[y]_{-x}^{2} = c\,[2+x] \\
&= \frac{1}{8}\,[2+x]
\end{aligned}$$

$$\begin{aligned} f_Y(y) &= c\int_{-y}^{2} dx \\ &= c\,[x]_{-y}^{2} = c\,[2+y] \\ &= \frac{1}{8}\,[2+y] \end{aligned}$$

$$f_X(x)f_Y(y) = \frac{1}{8}\,[2+x]\,\frac{1}{8}\,[2+y]$$

e.

$$f_X(x)f_Y(y) \neq f_{XY}(xy)$$

Therefore X and Y are independent.

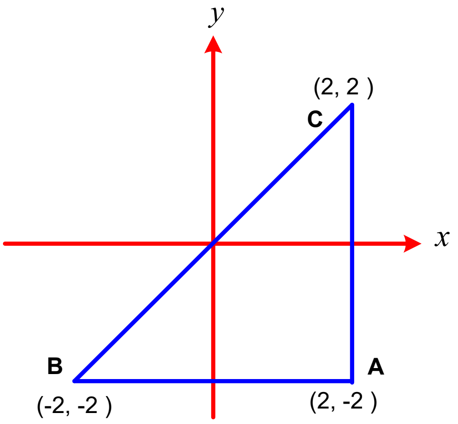
13 As shown in Figure is a region in the x, y plane where the bivariate pdf $f_{XY}(xy) = c$. Elsewhere the pdf is 0.

[a.] **What value must c have?**

[b.] **Evaluate $F_{XY}(1,1)$**

[c.] **Find the pdfs $f_X(x)$ and $f_Y(y)$.**

[d.] **Are X and Y independent ?** [?]



Integration Limits

By taking line BC. Considering x varies from -2 to 2 and y is a variable its lower limit is -2 and its upper limit is

$x_1 = -2,\ y_1 = -2,\ x_2 = 2,\ y_2 = 2$

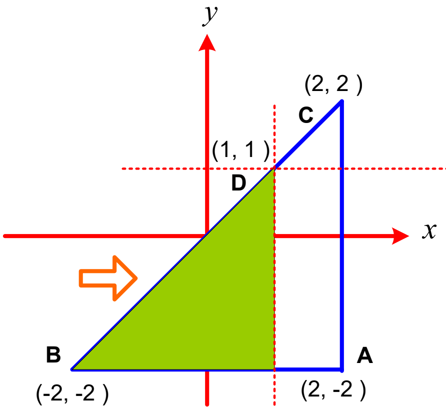
$$\begin{aligned} y - y_1 &= \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \\ y - (-2) &= \frac{2-(-2)}{2-(-2)}(x-(-2)) \\ y + 2 &= x + 2 \\ y &= x \end{aligned}$$

Solution:

a.

$$\begin{aligned}
F_{XY}(2,2) &= \int_{-2}^{2}\int_{-2}^{x} c\ dydx \\
1 &= c\int_{-2}^{2}\left[\int_{-2}^{x} dy\right] dx \\
1 &= c\int_{-2}^{2} [y]_{-2}^{x}\, dx = c\int_{-2}^{2}[x+2]dx = c\left[\frac{x^2}{2}+2x\right]_{-2}^{2} \\
&= \frac{c}{2}\left[x^2+4x\right]_{-2}^{2} = \frac{c}{2}\left[[(2)^2+4\times 2]-[(-2)^2+4\times(-2)]\right] \\
&= \frac{c}{2}[[4+8]-[4-8]] = \frac{c}{2}[12-4] \\
1 &= 8c \\
c &= \frac{1}{8}
\end{aligned}$$

b.



Integration Limits

By taking line BD. Considering x varies from -2 to 1 and y is a variable its lower limit is -2 and its upper limit is

$x_1 = -2,\ y_1 = -2,\ x_2 = 1,\ y_2 = 1$

$$\begin{aligned}
y - y_1 &= \frac{y_2-y_1}{x_2-x_1}(x-x_1) \\
y-(-2) &= \frac{1-(-2)}{1-(-2)}(x-(-2)) \\
y+2 &= x+2 \\
y &= x
\end{aligned}$$

$$\begin{aligned}
F_{XY}(1,1) &= \int_{-2}^{1}\int_{-2}^{x} c\ dydx \\
&= c\int_{-2}^{1}\left[\int_{-2}^{x} dy\right] dx \\
&= c\int_{-2}^{1} [y]_{-2}^{x}\, dx = c\int_{-2}^{1}[x+2]dx = c\left[\frac{x^2}{2}+x\right]_{-2}^{1} dx \\
&= \frac{c}{2}\left[x^2+4x\right]_{-2}^{1} = \frac{c}{2}\left[[(1)^1+4\times 1]-[(-2)^2+4\times(-2)+]\right] \\
&= \frac{c}{2}[[1+4]-[4-8]] = \frac{c}{2}[5+4] \\
&= \frac{9}{16}
\end{aligned}$$

c.

$$\begin{aligned}
f_X(x) &= c\int_{-2}^{x} dy \\
&= c\,[y]_{-2}^{x} = c\,[x+2] \\
&= \frac{1}{8}\,[x+2]
\end{aligned}$$

$$\begin{aligned}
f_Y(y) &= c\int_{y}^{2} dx \\
&= c\,[x]_{y}^{2} = c\,[2-y] \\
&= \frac{1}{8}\,[2-y]
\end{aligned}$$

$$f_X(x)f_Y(y) = \frac{1}{8}[x+2]\frac{1}{8}[2-y]$$

e.

$$f_X(x)f_Y(y) \neq f_{XY}(xy)$$

Therefore X and Y are independent.

14 A bivariate random variable has the following cdf.

$$F_{XY}(xy) = c(x+1)^2(y+1)^2 \quad (-1 < x < 4) \quad and \quad (-1 < y < 2)$$

outside of the given intervals, the bivariate cdf is as required by theory

[a.] **What value must c have?**

[b.] **Find the bivariate pdf**

[c.] **Find the cdfs** $F_X(x)$ **and** $F_Y(y)$**.**

[d.] **Evaluate** $P\{(X \leq 2) \cap (Y \leq 1)\}$

[e.] **Are the bivariate random variables independent ?** [?]

Solution:

a.

$$\begin{aligned} F_{XY}(4,2) &= c(x+1)^2(y+1)^2 \\ 1 &= c(4+1)^2(2+1)^2 = (25)(9) \\ 1 &= c225 \\ c &= \frac{1}{225} \end{aligned}$$

b. Bivariate pdf

$$\begin{aligned} \frac{\partial^2}{\partial x \partial y} c(x+1)^2(y+1)^2 &= c4(x+1)(y+1) \\ &= \frac{4}{225}(x+1)(y+1) \end{aligned}$$

c. The cdfs $F_X(x)$ and $F_Y(y)$.

$$\begin{aligned} F_X(x) &= F_{XY}(x,\infty) = F_{XY}(x,2) \\ &= c(x+1)^2(2+1)^2 \\ &= \frac{9}{225}(x+1)^2 \qquad (-1 < x < 4) \end{aligned}$$

$$\begin{aligned} F_Y(y) &= F_{XY}(\infty,y) = F_{XY}(2,y) \\ &= c(2+1)^2(y+1)^2 \\ &= \frac{25}{225}(y+1)^2 \qquad (-1 < y < 2) \end{aligned}$$

d. $P\{(X \leq 2) \cap (Y \leq 1)\}$

$$\begin{aligned} P\{(X \leq 2) \cap (Y \leq 1)\} &= F_{XY}(2,1) \\ &= c(x+1)^2(y+1)^2 \\ &= c(2+1)^2(1+1)^2 \\ &= \frac{9 \times 4}{225} \\ &= \frac{4}{25} \end{aligned}$$

e.

$$\begin{aligned} F_{XY}(xy) &= c(x+1)^2(y+1)^2 \\ F_X(x)F_Y(y) &= \frac{9}{225}(x+1)^2\frac{25}{225}(y+1)^2 \\ &= \frac{1}{225}(x+1)^2(y+1)^2 \\ F_X(x)F_Y(y) &= F_{XY}(xy) \end{aligned}$$

Therefore X and Y are independent.

15 A bivariate random variable has the following cdf.

$$F_{XY}(xy) = c(x+1)^2(y+1)^2 \quad (-1 < x < 3) \quad and \quad (-1 < y < 4)$$

outside of the given intervals, the bivariate cdf is as required by theory

[a.] **What value must c have?**

[b.] **Find the bivariate pdf**

[c.] **Find the cdfs** $F_X(x)$ **and** $F_Y(y)$**.**

[d.] **Evaluate** $P\{(X \leq 2) \cap (Y \leq 1)\}$

[e.] **Are the bivariate random variables independent ?** [?]

Solution:

a.

$$\begin{aligned} F_{XY}(3,4) &= c(x+1)^2(y+1)^2 \\ 1 &= c(3+1)^2(4+1)^2 = (16)(25) \\ 1 &= c400 \\ c &= \frac{1}{400} \end{aligned}$$

b. Bivariate pdf

$$\begin{aligned} \frac{\partial^2}{\partial x \partial y} c(x+1)^2(y+1)^2 &= c4(x+1)(y+1) \\ &= \frac{1}{100}(x+1)(y+1) \end{aligned}$$

c. The cdfs $F_X(x)$ and $F_Y(y)$.

$$\begin{aligned} F_X(x) &= F_{XY}(x,\infty) = F_{XY}(x,4) \\ &= c(x+1)^2(4+1)^2 \\ &= \frac{25}{400}(x+1)^2 \qquad (-1 < x < 3) \end{aligned}$$

$$\begin{aligned} F_Y(y) &= F_{XY}(\infty, y) = F_{XY}(3, y) \\ &= c(3+1)^2(y+1)^2 \\ &= \frac{16}{400}(y+1)^2 \qquad (-1 < y < 2) \end{aligned}$$

d. $P\{(X \leq 2) \cap (Y \leq 1)\}$

$$\begin{aligned} P\{(X \leq 2) \cap (Y \leq 1)\} &= F_{XY}(2, 1) \\ &= c(x+1)^2(y+1)^2 \\ &= c(2+1)^2(1+1)^2 \\ &= \frac{9 \times 4}{400} \\ &= \frac{9}{100} \end{aligned}$$

e.

$$\begin{aligned} F_{XY}(xy) &= c(x+1)^2(y+1)^2 \\ F_X(x)F_Y(y) &= \frac{25}{400}(x+1)^2\frac{16}{400}(y+1)^2 \\ &= \frac{1}{400}(x+1)^2(y+1)^2 \\ F_X(x)F_Y(y) &= F_{XY}(xy) \end{aligned}$$

Therefore X and Y are independent.

16 A bivariate random variable has the following cdf.

$$F_{XY}(xy) = c(x+1)^2(y+1)^2 \quad (-1 < x < 3) \quad and \quad (-1 < y < 2)$$

outside of the given intervals, the bivariate cdf is as required by theory

[a.] **What value must c have?**

[b.] **Find the bivariate pdf**

[c.] **Find the cdfs** $F_X(x)$ **and** $F_Y(y)$**.**

[d.] **Evaluate** $P\{(X \leq 2) \cap (Y \leq 1)\}$

[e.] **Are the bivariate random variables independent ?** [?]

Solution:

a.

$$\begin{aligned} F_{XY}(3, 2) &= c(x+1)^2(y+1)^2 \\ 1 &= c(3+1)^2(2+1)^2 = (16)(9) \\ 1 &= c144 \\ c &= \frac{1}{144} \end{aligned}$$

b. Bivariate pdf

$$\begin{aligned} \frac{\partial^2}{\partial x \partial y}c(x+1)^2(y+1)^2 &= c4(x+1)(y+1) \\ &= \frac{4}{144}(x+1)(y+1) \end{aligned}$$

c. The cdfs $F_X(x)$ and $F_Y(y)$.

$$\begin{aligned} F_X(x) &= F_{XY}(x,\infty) = F_{XY}(x,2) \\ &= c(x+1)^2(2+1)^2 \\ &= \frac{9}{144}(x+1)^2 \qquad (-1 < x < 3) \end{aligned}$$

$$\begin{aligned} F_Y(y) &= F_{XY}(\infty,y) = F_{XY}(3,y) \\ &= c(3+1)^2(y+1)^2 \\ &= \frac{16}{144}(y+1)^2 \qquad (-1 < y < 2) \end{aligned}$$

d. $P\{(X \le 2) \cap (Y \le 1)\}$

$$\begin{aligned} P\{(X \le 2) \cap (Y \le 1)\} &= F_{XY}(2,1) \\ &= c(x+1)^2(y+1)^2 \\ &= c(2+1)^2(1+1)^2 \\ &= \frac{9 \times 4}{144} \\ &= \frac{36}{144} \end{aligned}$$

e.

$$\begin{aligned} F_{XY}(xy) &= c(x+1)^2(y+1)^2 \\ F_X(x)F_Y(y) &= \frac{9}{144}(x+1)^2\frac{16}{144}(y+1)^2 \\ &= \frac{1}{144}(x+1)^2(y+1)^2 \\ F_X(x)F_Y(y) &= F_{XY}(xy) \end{aligned}$$

Therefore X and Y are independent.

Note: Entire material is taken from different text books or from the Internet (different websites). Slightly it is modified from the original content. It is not for any commercial purpose. It is used to teach students. Suggestions are always encouraged.

1.2 Bivariate-Expectations

The expectation operation to a continuous random variables X and Y, is defined as:

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y)dxdy$$

where $g(x,y)$ is an arbitrary function of two variables. If $g(x,y)$ is of only single random variable x then

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\int_{-\infty}^{\infty} f_{XY}(x,y)dydx = \int_{-\infty}^{\infty} g(x) f_X(x)dx$$

The correlation of X and Y is the expected value of the product of X and Y

$$E[X,Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{XY}(x,y)dxdy$$

The expectation is also same as averaging, therefore

$$E[X,Y] \sim \frac{1}{n}\sum_{i=1}^{n} x_i y_i$$

Properties of correlation

1. Positive correlation: If the product tends to positive i.e.,

$$\frac{1}{n}\sum_{i=1}^{n} x_i y_i > 0$$

2. Negative correlation: If the product tends to negative i.e.,

$$\frac{1}{n}\sum_{i=1}^{n} x_i y_i < 0$$

3. uncorrelation: If the product tends to

$$\frac{1}{n}\sum_{i=1}^{n} x_i y_i = 0$$

then it is said X and Y are uncorrelated with each other

If the bivariate random variables do not have means of 0 then correlation is defined as covariance denoted as $Cov[XY]$ and is expressed as

$$\begin{aligned} Cov[XY] &= E[(X-\mu_X)(Y-\mu_Y)] \\ &= E[XY - \mu_X Y - \mu_Y X + \mu_X\mu_Y] \\ &= E[XY] - \mu_X E[Y] - \mu_Y E[X] + \mu_X\mu_Y] \\ &= E[XY] - \mu_X\mu_Y] \end{aligned}$$

Uncorrelated X and Y

$$Cov[XY] = 0$$

then X and Y are uncorrelated with each other

$$E[XY] = \mu_X\mu_Y$$

Orthogonal X and Y

$$Cov[XY] = 0$$

then X and Y are uncorrelated with each other

$$E[XY] = 0$$

$$Cov[XY] = -\mu_X\mu_Y$$

Correlated X and Y:

A correlation coefficient denoted ρ_{XY} is defined as

$$\rho_{XY} = \frac{Cov[XY]}{\sigma_X \sigma_Y}$$

$$E\left[\left(\frac{X-\mu_X}{\sigma_X} \pm \frac{Y-\mu_Y}{\sigma_Y}\right)^2\right] \geq 0$$

$$E\left[\left(\frac{X-\mu_X}{\sigma_X}\right)^2 \pm 2\frac{(X-\mu_X)(Y-\mu_Y)}{\sigma_X\sigma_Y} + \left(\frac{Y-\mu_Y}{\sigma_Y}\right)^2\right] \geq 0$$

$$1 \pm 2\rho_{xy} + 1 \geq 0$$

$$\rho_{xy} \leq 1$$

$$|\rho_{xy}| \pm 1$$

Consider a relation between X and Y is defined as

$$Y = aX + b$$

then

$$\begin{aligned} Cov[XY] &= E[(X-\mu_X)(aX+b-a\mu_X-b)] \\ &= E[(X-\mu_X)a(X-\mu_X)] \\ &= aE[(X-\mu_X)^2] \\ &= a\sigma_X^2 \end{aligned}$$

The standard deviation of Y is

$$\sigma_Y = \pm\sqrt{a^2}\sigma_X$$

$$\rho_{XY} = \frac{a\sigma_X^2}{\pm\sqrt{a^2}\sigma_X} = \pm 1$$

17. The mean and variance of random variable X are -2 and 3; the mean and variance of Y are 3 and 5. The covariance $Cov[XY] = -0.8$. What are the correlation coefficient ρ_{XY} and the correlation $E[XY]$? [?]

Solution:

a. Correlation coefficient ρ_{XY} is

$$\begin{aligned}\rho_{XY} &= \frac{Cov[XY]}{\sigma_X\sigma_Y}\\ &= \frac{-0.8}{\sqrt{3\times 5}}\\ &= -0.2066\end{aligned}$$

b.

$$\begin{aligned}E[XY] &= Cov[XY] + \mu_X\mu_Y\\ &= -0.8 + (-2)(3)\\ &= -6.8\end{aligned}$$

18. The mean and variance of random variable X are -2 and 3; the mean and variance of Y are 3 and 5. The correlation coefficient $\rho_{XY} = 0.7$. What are the $Cov[XY]$ and the correlation $E[XY]$? [?]

Solution:

a. Correlation coefficient ρ_{XY} is

$$\begin{aligned}cov_{XY} &= \rho_{XY}\sigma_X\sigma_Y\\ &= 0.7\sqrt{3\times 5}\\ &= 2.7111\end{aligned}$$

b.

$$\begin{aligned}E[XY] &= Cov_{XY} + \mu_X\mu_Y\\ &= 2.7111 + (-2)(3)\\ &= -3.2889\end{aligned}$$

19. The mean and variance of random variable X are -2 and 3; the mean and variance of Y are 3 and 5. The correlation $E[XY] = -8.7$. What are the $Cov[XY]$ and the correlation coefficient ρ_{XY}? [?]

Solution:

a. Correlation coefficient ρ_{XY} is

$$\begin{aligned}Cov_{XY} &= E[XY] - \mu_X\mu_Y\\ &= -8.7 - (-2)(3)\\ &= -2.7\end{aligned}$$

b.

$$\begin{aligned}\rho_{XY} &= \frac{Cov_{XY}}{\sigma_X\sigma_Y}\\ &= \frac{-2.7}{\sqrt{(3)(5)}}\\ &= -0.6971\end{aligned}$$

20. X is random variable $\mu_X = 4$ $\sigma_X = 5$. Y is a random variable, $\mu_Y = 6$ $\sigma_Y = 7$. The correlation coefficient is -0.7. If $U = 3X + 2Y$. What are the $Var[U]$, $Cov[UX]$, $Cov[UY]$? [?]

Solution:

a. $Var[U]$

$$\begin{aligned}Cov_{XY} &= \rho_{XY}\sigma_X\sigma_Y\\ &= (-0.7)(5)(7)\\ &= -24.5\end{aligned}$$

$$\begin{aligned}\sigma_U^2 &= E[(U-\mu_U)^2]\\ &= E[9(X-\mu_X)^2 + 12(X-\mu_X)(Y-\mu_Y) + 4(Y-\mu_Y)]\\ &= 9\sigma_X^2 + 12Cov[XY] + 4\sigma_Y^2\\ &= 9\times(5^2) + 12(-24.5) + 4(7^2)\\ &= 225 - 294 + 196\\ &= 127\end{aligned}$$

b. $Cov[UX]$

$$\begin{aligned} Cov[UX] &= E[(U-\mu_U)(X-\mu_X)] \\ &= E[\{3(X-\mu_X)+2(Y-\mu_Y)\}(X-\mu_X)] \\ &= 3\sigma_X^2 + 2Cov[XY] \\ &= 3(5^2)+2(-24.5) = 75-49 \\ &= 26 \end{aligned}$$

c. $Cov[UY]$

$$\begin{aligned} Cov[UY] &= E[(U-\mu_U)(Y-\mu_Y)] \\ &= E[\{3(X-\mu_X)+2(Y-\mu_Y)\}(Y-\mu_Y)] \\ &= 3Cov[XY] + 2\sigma_Y^2 \\ &= 3(-24.5)+2(7^2) = -73.5+98 \\ &= 24.5 \end{aligned}$$

21. X is random variable $\mu_X = 4$ $\sigma_X = 5$. Y is a random variable, $\mu_Y = 6$ $\sigma_Y = 7$. The correlation coefficient is 0.2. If $U = 3X + 2Y$. What are the $Var[U]$, $Cov[UX]$ and $Cov[UY]$? [?]

Solution:

a. $Var[U]$

$$\begin{aligned} Cov_{XY} &= \rho_{XY}\sigma_X\sigma_Y \\ &= (0.2)(5)(7) \\ &= 7 \end{aligned}$$

$$\begin{aligned} \sigma_U^2 &= E[(U-\mu_U)^2] \\ &= E[9(X-\mu_X)^2 + 12(X-\mu_X)(Y-\mu_Y) + 4(Y-\mu_Y)] \\ &= 9\sigma_X^2 + 12Cov[XY] + 4\sigma_Y^2 \\ &= 9\times(5^2)+12(7)+4(7^2) \\ &= 225+84+196 \\ &= 505 \end{aligned}$$

b. $Cov[UX]$

$$\begin{aligned} Cov[UX] &= E[(U-\mu_U)(X-\mu_X)] \\ &= E[\{3(X-\mu_X)+2(Y-\mu_Y)\}(X-\mu_X)] \\ &= 3\sigma_X^2 + 2Cov[XY] \\ &= 3(5^2)+2(7) = 75+14 \\ &= 89 \end{aligned}$$

c. $Cov[UY]$

$$\begin{aligned} Cov[UY] &= E[(U-\mu_U)(Y-\mu_Y)] \\ &= E[\{3(X-\mu_X)+2(Y-\mu_Y)\}(Y-\mu_Y)] \\ &= 3Cov[XY] + 2\sigma_Y^2 \\ &= 3(7)+2(7^2) = 21+98 \\ &= 119 \end{aligned}$$

22. X is random variable $\mu_X = 4$ $\sigma_X = 5$. Y is a random variable, $\mu_Y = 6$ $\sigma_Y = 7$. The correlation coefficient is 0.7. If $U = 3X + 2Y$. What are the $Var[U]$, $Cov[UX]$ and $Cov[UY]$? [?]

Solution:

a. $Var[U]$

$$\begin{aligned} Cov_{XY} &= \rho_{XY}\sigma_X\sigma_Y \\ &= (0.7)(5)(7) \\ &= 24.5 \end{aligned}$$

$$\begin{aligned} \sigma_U^2 &= E[(U-\mu_U)^2] \\ &= E[9(X-\mu_X)^2 + 12(X-\mu_X)(Y-\mu_Y) + 4(Y-\mu_Y)] \\ &= 9\sigma_X^2 + 12Cov[XY] + 4\sigma_Y^2 \\ &= 9 \times (5^2) + 12(24.5) + 4(7^2) \\ &= 225 + 294 + 196 \\ &= 715 \end{aligned}$$

b. $Cov[UX]$

$$\begin{aligned} Cov[UX] &= E[(U-\mu_U)(X-\mu_X)] \\ &= E[\{3(X-\mu_X) + 2(Y-\mu_Y)\}(X-\mu_X)] \\ &= 3\sigma_X^2 + 2Cov[XY] \\ &= 3(5^2) + 2(24.5) = 75 + 49 \\ &= 124 \end{aligned}$$

c. $Cov[UY]$

$$\begin{aligned} Cov[UY] &= E[(U-\mu_U)(Y-\mu_Y)] \\ &= E[\{3(X-\mu_X) + 2(Y-\mu_Y)\}(Y-\mu_Y)] \\ &= 3Cov[XY] + 2\sigma_Y^2 \\ &= 3(24.5) + 2(7^2) = 73.5 + 98 \\ &= 171.5 \end{aligned}$$

23. X and Y are correlated random variable with a correlation coefficient of $\rho = 0.6$ $\mu_X = 3$ $Var[X] = 49$, $\mu_Y = 144$ $Var[Y] = 144$. The random variables U and V are obtained using $U = X + cY$ and $V = X - cY$. What values can c have if U and V are uncorrelated? [?]

Solution:

$$\begin{aligned} Cov[UV] &= E[(U-\mu_U)(V-\mu_V)] \\ &= E[((X-\mu_X) + c(Y-\mu_Y))((X-\mu_X) - c(Y-\mu_Y))] \\ &= \sigma_X^2 - c^2\sigma_Y^2 \end{aligned}$$

If $Cov[UV] = 0$ then

$$\begin{aligned} \sigma_X^2 - c^2\sigma_Y^2 &= 0 \\ c &= \pm\frac{\sigma_X}{\sigma_Y} \\ &= \pm\sqrt{\frac{49}{144}} \\ &= \pm 0.5833 \end{aligned}$$

24. **X and Y are correlated random variable with a correlation coefficient of $\rho = 0.7$ $\mu_X = 5$ $Var[X] = 36$, $\mu_Y = 16$ $Var[Y] = 150$. The random variables U and V are obtained using $U = X + cY$ and $V = X - cY$. What values can c have if U and V are uncorrelated?** [?]

Solution:

$$\begin{aligned}
Cov[UV] &= E[(U - \mu_U)(V - \mu_V)] \\
&= E[((X - \mu_X) + c(Y - \mu_Y))((X - \mu_X) - c(Y - \mu_Y))] \\
&= \sigma_X^2 - c^2\sigma_Y^2
\end{aligned}$$

If $Cov[UV] = 0$ then

$$\begin{aligned}
\sigma_X^2 - c^2\sigma_Y^2 &= 0 \\
c &= \pm\frac{\sigma_X}{\sigma_Y} \\
&= \pm\sqrt{\frac{36}{150}} \\
&= \pm 0.4899
\end{aligned}$$

25. **X and Y are correlated random variable with a correlation coefficient of $\rho = 0.8$ $\mu_X = 20$ $Var[X] = 70$, $\mu_Y = 15$ $Var[Y] = 100$. The random variables U and V are obtained using $U = X + cY$ and $V = X - cY$. What values can c have if U and V are uncorrelated?** [?]

Solution:

$$\begin{aligned}
Cov[UV] &= E[(U - \mu_U)(V - \mu_V)] \\
&= E[((X - \mu_X) + c(Y - \mu_Y))((X - \mu_X) - c(Y - \mu_Y))] \\
&= \sigma_X^2 - c^2\sigma_Y^2
\end{aligned}$$

If $Cov[UV] = 0$ then

$$\begin{aligned}
\sigma_X^2 - c^2\sigma_Y^2 &= 0 \\
c &= \pm\frac{\sigma_X}{\sigma_Y} \\
&= \pm\sqrt{\frac{70}{100}} \\
&= \pm 0.8367
\end{aligned}$$

Note: Entire material is taken from different text books or from the Internet (different websites). Slightly it is modified from the original content. It is not for any commercial purpose. It is used to teach students. Suggestions are always encouraged.

1.3 Bivariate Transformations

- Consider a bivariate random variables X and Y with known mean, variance and their covariance are transformed to U and V with linear transformation is as follows.

$$\begin{aligned} U &= aX + bY \\ V &= cX + dY \end{aligned}$$

Then the means of U and V are

$$\begin{aligned} \mu_U &= a\mu_X + b\mu_Y \\ \mu_V &= c\mu_X + d\mu_Y \end{aligned}$$

The variance of U is

$$\begin{aligned} \sigma_U^2 &= E[(U - \mu_U)^2] \\ &= E[(aX + bY - a\mu_X - b\mu_Y)^2] \\ &= E[(a(X - \mu_X) + b(Y - \mu_Y))^2] \\ &= E[a^2(X - \mu_X)^2 + 2ab(X - \mu_X)(Y - \mu_Y) + b^2(Y - \mu_Y)^2] \\ &= a^2\sigma_X^2 + 2abCov[XY] + b^2\sigma_Y^2 \end{aligned}$$

Similarly the variance of V is

$$\begin{aligned} \sigma_V^2 &= E[(V - \mu_V)^2] \\ &= E[(cX + dY - cmu_X - d\mu_Y)^2] \\ &= E[(c(X - \mu_X) + d(Y - \mu_Y))^2] \\ &= E[c^2(X - \mu_X)^2 + 2cd(X - \mu_X)(Y - \mu_Y) + d^2(Y - \mu_Y)^2] \\ &= c^2\sigma_X^2 + 2cdCov[XY] + d^2\sigma_Y^2 \end{aligned}$$

$$Cov[UV] = ac\sigma_X^2 + (bc + ad)Cov[XY] + bd\sigma_Y^2$$

$$\begin{aligned} U &= cos\theta X - sin\theta Y \\ V &= sin\theta X + cos\theta Y \end{aligned}$$

The inverse of the rotational transformations is

$$\begin{aligned} X &= cos\theta U + sin\theta V \\ Y &= -sin\theta U + cos\theta V \end{aligned}$$

Then the means of X and Y are

$$\begin{aligned} \mu_X &= cos\theta\mu_U + sin\theta\mu_V \\ \mu_Y &= -sin\theta\mu_U + cos\theta\mu_V \end{aligned}$$

$$\begin{aligned} \sigma_X^2 &= cos^2\theta\sigma_U^2 + 2sin\theta cos\theta Cov[UV] + sin^2\theta\sigma_V^2 \\ \sigma_Y^2 &= sin^2\theta\sigma_U^2 - 2sin\theta cos\theta Cov[UV] + cos^2\theta\sigma_V^2 \\ Cov[XY] &= sin\theta cos\theta[\sigma_V^2 - \sigma_U^2] + (cos^2\theta - sin^2\theta)Cov[UV] \end{aligned}$$

26. The zero mean bivariate random variables X_1 and X_2 have the following variances: $Var[X_1] = 2$ and $Var[X_2] = 4$. Their correlation coefficient is 0.8. Random variables Y_1 and Y_2 are obtained from

$$\begin{aligned} Y_1 &= 3X_1 + 4X_2 \\ Y_2 &= -X_1 + 2X_2 \end{aligned}$$

Find values of $Var[Y_1]$ and $Var[Y_2]$ and $Cov[Y_1 Y_2]$ [?]

Solution:

$$\begin{aligned} Cov[X_1 X_2] &= \rho_{X1X2}\sigma_{X_1}\sigma_{X_2} \\ &= (0.8)\sqrt{2 \times 4} \\ &= 2.2627 \end{aligned}$$

$$\begin{aligned} \sigma_{Y_1}^2 &= a^2\sigma_{X_1}^2 + 2abCov[X_1 X_2] + b^2\sigma_{X_2}^2 \\ &= (3)^2(2) + 2(3)(4)(2.2627) + (4^2)4 \\ &= 136.3058 \end{aligned}$$

$$\begin{aligned} \sigma_{Y_2}^2 &= c^2\sigma_{X_1}^2 + 2cdCov[X_1 X_2] + d^2\sigma_{X_2}^2 \\ &= (-1)^2(2) + 2(-1)(2)(2.2627) + (2)^2 4 \\ &= 8.9492 \end{aligned}$$

$$\begin{aligned} Cov[Y_1 Y_2] &= ac\sigma_{X_1}^2 + (bc + ad)Cov[X_1 X_2] + bd\sigma_{X_2}^2 \\ &= (3)(-1)(2) + [(4)(-1) + (3)(2)](2.2627) + (4)(2)(4) \\ &= 30.5254 \end{aligned}$$

27. The random variable X has a mean of 3.0 and variances: of 0.7. The random variable Y has a mean of -3.0 and variance of 0.6. The covariances for X and Y is 0.4666. Given the transformation

$$\begin{aligned} U &= 10X + 6Y \\ V &= 5X + 13Y \end{aligned}$$

Calculate the values of $Var[U]$ and $Var[V]$ and $Cov[UV]$ [?]

Solution:

Given $Cov[XY] = 0.4666$

$$\begin{aligned} \sigma_U^2 &= a^2\sigma_X^2 + 2abCov[XY] + b^2\sigma_Y^2 \\ &= (10)^2(0.7) + 2(10)(6)(0.4666) + (6^2)(0.6) \\ &= 147.5920 \end{aligned}$$

$$\begin{aligned} \sigma_V^2 &= a^2\sigma_X^2 + 2abCov[XY] + b^2\sigma_Y^2 \\ &= (5)^2(0.7) + 2(5)(13)(0.4666) + (13^2)(0.6) \\ &= 179.5580 \end{aligned}$$

$$\begin{aligned} Cov[UV] &= ac\sigma_{X_1}^2 + (bc + ad)Cov[X_1 X_2] + bd\sigma_{X_2}^2 \\ &= (10)(5)(0.7) + [(6)(5) + (10)(13)](0.4666) + (6)(13)(0.6) \\ &= 156.4560 \end{aligned}$$

28. The random variables U and V are related to X and Y with

$$\begin{aligned} U &= 2X - 3Y \\ V &= -4X + 2Y \end{aligned}$$

We know that $\mu_X = 13,\ \mu_Y = -7,\ \sigma_X^2 = 5,\ \sigma_Y^2 = 6$ and $Cov[XY] = 0$ Calculate values for $Var[U]$ and $Var[V]$ and $Cov[UV]$ [?]

Solution:

$$\begin{aligned} \sigma_U^2 &= a^2\sigma_X^2 + 2abCov[XY + b^2\sigma_Y^2 \\ &= (2)^2(5) + 0 + ((-3)^2)(6) \\ &= 74 \end{aligned}$$

$$\begin{aligned} \sigma_V^2 &= a^2\sigma_X^2 + 2abCov[XY + b^2\sigma_Y^2 \\ &= (-4)^2(5) + 0 + (2^2)(6) \\ &= 104 \end{aligned}$$

$$\begin{aligned} Cov[UV] &= ac\sigma_{X_1}^2 + (bc + ad)Cov[X_1X_2] + bd\sigma_{X_2}^2 \\ &= (2)(-4)(5) + 0 + (-3)(2)(6) \\ &= -76 \end{aligned}$$

29. It is required to have correlated bivariate random variables U and V such that $\mu_U = 0,\ \mu_V = 0,\ \sigma_U^2 = 7,\ \sigma_V^2 = 20$ and $\rho_{UV} = 0.50$. Specify uncorrelated random variables X and Y and an angle θ, that when used in the transformation $U = cos\theta X - sin\theta Y$, $V = sin\theta X + cos\theta Y$ will produce the desired U and V. [?]

Solution:

$$\begin{aligned} \mu_X &= a\mu_U + b\mu_V = 0 + 0 = 0 \\ \mu_Y &= c\mu_U + d\mu_V = 0 + 0 = 0 \end{aligned}$$

$$\begin{aligned} Cov[UV] &= \rho_{UV}\sigma_U\sigma_V \\ &= 0.5\sqrt{(7)(20)} \\ &= 5.9161 \end{aligned}$$

$$\begin{aligned} tan2\theta &= \frac{2Cov[UV]}{\sigma_U^2 - \sigma_V^2} \\ &= \frac{2(5.9161)}{7 - 20} = -0.9101 \\ 2\theta &= tan^{-1}(-0.9101) = -42.3055 \\ \theta &= -21.1537 \end{aligned}$$

$$\begin{aligned} cos\theta &= cos(-21.1537) = 0.9754 \\ sin\theta &= sin(-21.1537) = -0.3609 \end{aligned}$$

$$\begin{aligned}
\sigma_X^2 &= cos^2\theta\sigma_U^2 + 2sin\theta cos\theta Cov[UV] + sin^2\theta\sigma_V^2 \\
&= (0.9754)^2(7) + 2(-0.3609)(0.9754)(5.9161) + (-0.3609)^2(20) \\
&= 6.6598 - 4.1651 + 2.6049 \\
&= 4.7107
\end{aligned}$$

$$\begin{aligned}
\sigma_Y^2 &= sin^2\theta\sigma_U^2 - 2sin\theta cos\theta Cov[UV] + cos^2\theta\sigma_V^2 \\
&= (-0.3609)^2(7) - 2(-0.3609)(0.9754)(5.9161) + (0.9754)^2(20) \\
&= 0.1302(7) - 2(-0.3609)(0.9326)(5.9161) + (0.9514)(20) \\
&= 0.9114 + 4.1651 + 19.0281 \\
&= 24.1046
\end{aligned}$$

30. It is required to have correlated bivariate random variables U and V such that $\mu_U = 0,\ \mu_V = 0,\ \sigma_U^2 = 25,\ \sigma_V^2 = 4$ and $\rho_{UV} = -0.50$. Specify uncorrelated random variables X and Y and an angle θ, that when used in the transformation $U = cos\theta X - sin\theta Y$, $V = sin\theta X + cos\theta Y$ will produce the desired U and V. [?]

Solution:

$$\begin{aligned}
\mu_X &= a\mu_U + b\mu_V = 0 + 0 = 0 \\
\mu_Y &= c\mu_U + d\mu_V = 0 + 0 = 0
\end{aligned}$$

$$\begin{aligned}
Cov[UV] &= \rho_{UV}\sigma_U\sigma_V \\
&= -0.5\sqrt{(25)(4)} \\
&= -5
\end{aligned}$$

$$\begin{aligned}
tan2\theta &= \frac{2Cov[UV]}{\sigma_U^2 - \sigma_V^2} \\
&= \frac{2(-5)}{25-4} = -0.4762 \\
2\theta &= tan^{-1}(-0.4762) = -25.4637 \\
\theta &= -12.7319
\end{aligned}$$

$$\begin{aligned}
cos\theta &= cos(-12.7319) = 0.9754 \\
sin\theta &= sin(-12.7319) = -0.2204
\end{aligned}$$

$$\begin{aligned}
\sigma_X^2 &= cos^2\theta\sigma_U^2 + 2sin\theta cos\theta Cov[UV] + sin^2\theta\sigma_V^2 \\
&= cos^2(-25.4637)(25) + 2(sin(-25.4637))cos(-25.4637))(-5) + (sin^2(-25.4637))(4) \\
&= 0.9514(25) + 2(-0.2204)(0.9754)(-5) + (0.1302)(4) \\
&= 23.7851 + 2.1497 + 0.1943 \\
&= 26.1292
\end{aligned}$$

$$
\begin{aligned}
\sigma_Y^2 &= sin^2\theta\sigma_U^2 - 2sin\theta cos\theta Cov[UV] + cos^2\theta\sigma_V^2 \\
&= (-0.2204)^2(25) - 2(-0.2204)(0.9754)(-5) + (0.9754)^2)(4) \\
&= 0.0485(25) - 2.1497 + (0.9166)(4) \\
&= 1.2125 - 2.1497 + 3.6664 \\
&= 2.7292
\end{aligned}
$$

31. It is required to have correlated bivariate random variables U and V such that $\mu_U = 0,\ \mu_V = 0,\ \sigma_U^2 = 7,\ \sigma_V^2 = 1$ and $\rho_{UV} = 0.30$. Specify uncorrelated random variables X and Y and an angle θ, that when used in the transformation $U = cos\theta X - sin\theta Y$, $V = sin\theta X + cos\theta Y$ will produce the desired U and V. [?]

Solution:

$$
\begin{aligned}
\mu_X &= a\mu_U + b\mu_V = 0 + 0 = 0 \\
\mu_Y &= c\mu_U + d\mu_V = 0 + 0 = 0
\end{aligned}
$$

$$
\begin{aligned}
Cov[UV] &= \rho_{UV}\sigma_U\sigma_V \\
&= 0.3\sqrt{(7)(1)} \\
&= 0.7937
\end{aligned}
$$

$$
\begin{aligned}
tan2\theta &= \frac{2Cov[UV]}{\sigma_U^2 - \sigma_V^2} \\
&= \frac{2(0.7937)}{7-1} = 0.2645 \\
2\theta &= tan^{-1}(0.2645) = 14.8154 \\
\theta &= 7.4077
\end{aligned}
$$

$$
\begin{aligned}
cos\theta &= cos(7.4077) = 0.9916 \\
sin\theta &= sin(-12.7319) = 0.1290
\end{aligned}
$$

$$
\begin{aligned}
\sigma_X^2 &= cos^2\theta\sigma_U^2 + 2sin\theta cos\theta Cov[UV] + sin^2\theta\sigma_V^2 \\
&= (0.9916)^2(7) + 2(0.1290)(0.9916)(0.7937) + (0.1290)^2(1) \\
&= 6.8828 + 0.2030 + 0.01644 \\
&= 7.1024
\end{aligned}
$$

$$
\begin{aligned}
\sigma_Y^2 &= sin^2\theta\sigma_U^2 - 2sin\theta cos\theta Cov[UV] + cos^2\theta\sigma_V^2 \\
&= (0.1290)^2(7) - 2(0.1290)(0.9916)(0.7937) + (0.9916)^2)(1) \\
&= 0.1164 - 0.2030 + 0.9832 \\
&= 0.8966
\end{aligned}
$$

Note: Entire material is taken from different text books or from the Internet (different websites). Slightly it is modified from the original content. It is not for any commercial purpose. It is used to teach students. Suggestions are always encouraged.

1.4 Sums of Two Independent Two Random Variables:

- Consider a two independent random variables X and Y and another random variable W is related as

$$W = X + Y$$

Then the mean and variance of W is

$$\begin{aligned} E[W] &= E[X+Y] \\ \mu_W &= \mu_X + \mu_Y \end{aligned}$$

The variance of W is

$$\begin{aligned} \sigma_W^2 &= E[(W-\mu_W)^2] \\ &= E[(X+Y-\mu_X-\mu_Y)^2] \\ &= E[((X-\mu_X)+(Y-\mu_Y))^2] \\ &= E[(X-\mu_X)^2 + 2(X-\mu_X)(Y-\mu_Y) + (Y-\mu_Y)^2] \\ &= \sigma_X^2 + 2Cov[XY] + \sigma_Y^2 \\ &= \sigma_X^2 + \sigma_Y^2 \end{aligned}$$

X and X are independent and are uncorrelated with each other hence $2Cov[XY]$
If pdf of X and Y are known then the cdf of the random variable W is

$$F_W(w) = P\{X+Y \leq w\}$$

The cdf for the random variable W is

$$\begin{aligned} P\{X+Y \leq w\} &= P\{(x,y) \in \Re\} \\ &= \int\int_{\Re} f_{XY}(x,y)dxdy \\ &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{w-x} f_{XY}(x,y)dy\right]dx \\ F_W(w) &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{w-x} f_{XY}(x,y)dy\right]dx \end{aligned}$$

The pdf for the random variable W is

$$f_W(w) = \int_{-\infty}^{\infty} f_{XY}(x, w-x)dx$$

Assuming that X and Y are independent then

$$\begin{aligned} f_W(w) &= \int_{-\infty}^{\infty} f_X(x)f_Y(w-x)dx \\ &= \int_{-\infty}^{\infty} f_Y(y)f_X(w-y)dy \end{aligned}$$

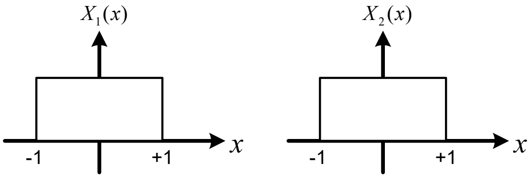
The above equation is convolution hence it can be written as

$$f_W(w) = f_X(x) * f_Y(y)$$

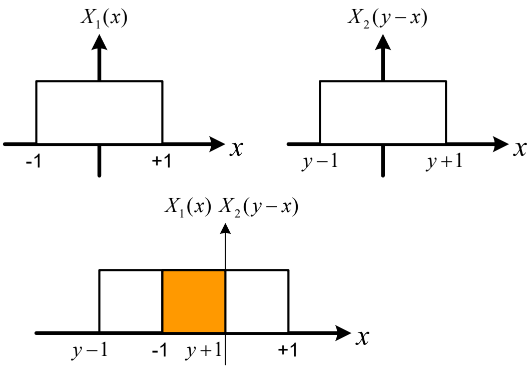
35. The random variables X is uniformly distributed between ± 1. Two independent realizations of are added: $Y = X_1 + X_2$. What is the pdf for Y [?]

Solution:

$$
\begin{aligned}
f_{X_1}(x) &= \frac{1}{b-a} = \frac{1}{1-(-1)} = \frac{1}{2} \\
f_{X_2}(y) &= \frac{1}{b-a} = \frac{1}{1-(-1)} = \frac{1}{2}
\end{aligned}
$$

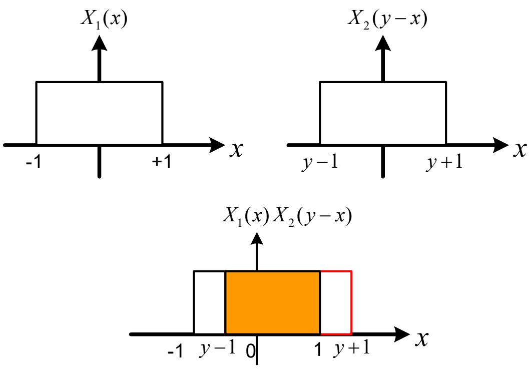


Case 1: $-1 < (y+1) < 1 \;\Rightarrow -2 < y < 0$



$$
\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f_X(x) f_X(y-x)dx \\
&= \int_{-1}^{y+1} \frac{1}{2} \times \frac{1}{2} dx \\
&= \frac{1}{4}[x]_{-1}^{y+1} = \frac{1}{4}[y+1-(-1)] \\
&= \frac{y+2}{4} \qquad -2 < y < 0
\end{aligned}
$$

Case 2: $-1 < (y-1) < 1 \;\Rightarrow 0 < y < 2$

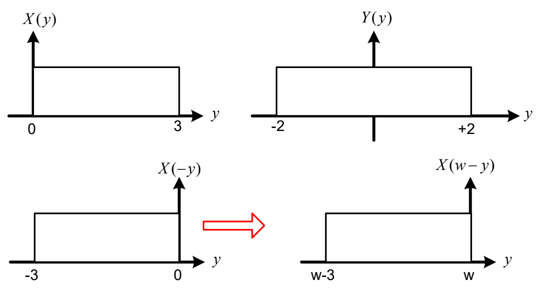


$$
\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f_X(x) f_X(y-x)dx \\
&= \int_{y-1}^{1} \frac{1}{2} \times \frac{1}{2} dx \\
&= \frac{1}{4}[x]_{y-1}^{1} = \frac{1}{4}[1-(y-1)] \\
&= \frac{2-y}{4} \qquad 0 < y < 2
\end{aligned}
$$

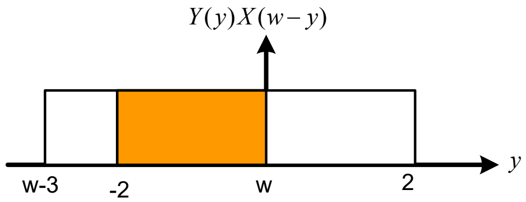
36. X is a random variable uniformly distributed between 0 and 3. Y is a random variable independent of X, uniformly distributed between +2 and -2. $W = X + Y$. What is the pdf for W [?]

Solution:

$$
\begin{aligned}
f_X(x) &= \frac{1}{b-a} = \frac{1}{3-0} = \frac{1}{3} \\
f_Y(y) &= \frac{1}{b-a} = \frac{1}{2-(-2)} = \frac{1}{4}
\end{aligned}
$$

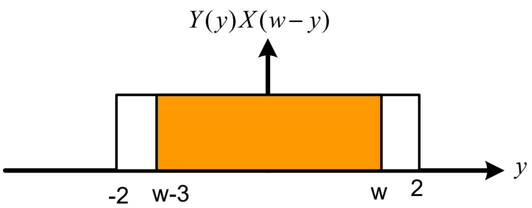


Case 1: Width of the window 3-0=3, Lower range=-2 upper range=-2+3=1 $\Rightarrow -2 < w < 1$



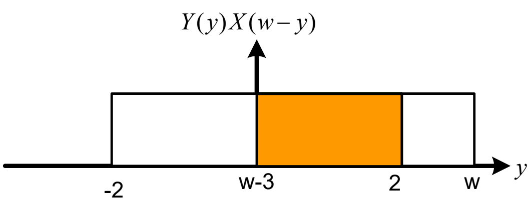
$$
\begin{aligned}
f_W(w) &= \int_{-\infty}^{\infty} f_Y(y) f_X(w-y)dy \\
&= \int_{-2}^{w} \frac{1}{4} \times \frac{1}{3} dy \\
&= \frac{1}{12}[y]_{-2}^{w} = \frac{1}{12}(w+2) \\
&= \frac{(w+2)}{12} \qquad -2 < w < 1
\end{aligned}
$$

Case 2: $1 < w < 2$



$$
\begin{aligned}
f_W(w) &= \int_{-\infty}^{\infty} f_Y(y) f_X(w-y)dy \\
&= \int_{w-3}^{w} \frac{1}{4} \times \frac{1}{3} dy \\
&= \frac{1}{12}[y]_{w-3}^{w} = \frac{1}{12}(w-(w-3)) \\
&= \frac{1}{4} \qquad 1 < w < 2
\end{aligned}
$$

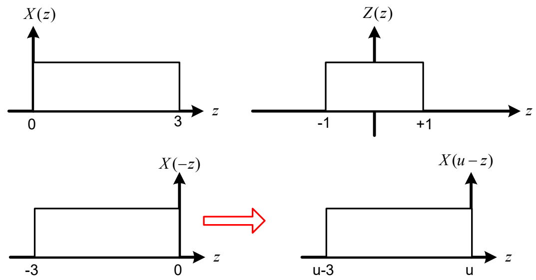
Case 3: Width of the window 3-0=3, Lower range=2 upper range=2+3=5 $\Rightarrow 2 < w < 5$



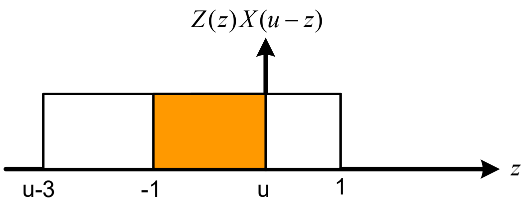
$$
\begin{aligned}
f_W(w) &= \int_{-\infty}^{\infty} f_Y(y) f_X(w-y)dy \\
&= \int_{w-3}^{2} \frac{1}{4} \times \frac{1}{3} dy \\
&= \frac{1}{12}[y]_{w-3}^{-2} = \frac{1}{12}(2-(w-3)) \\
&= \frac{5-w}{12} \qquad 2 < w < 5
\end{aligned}
$$

37. X is a random variable uniformly distributed between 0 and 3. Z is a random variable independent of X, uniformly distributed between +1 and -1. $U = X + Z$. What is the pdf for U [?] Solution:

$$
\begin{aligned}
f_X(x) &= \frac{1}{b-a} = \frac{1}{3-0} = \frac{1}{3} \\
f_Z(z) &= \frac{1}{b-a} = \frac{1}{1-(-1)} = \frac{1}{2}
\end{aligned}
$$

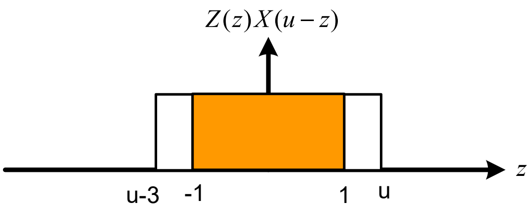


Case 1: $-1 < u < 1$



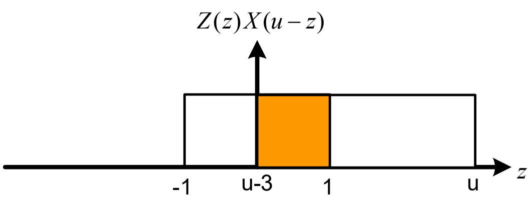
$$
\begin{aligned}
f_U(u) &= \int_{-\infty}^{\infty} f_Z(z) f_X(u-z)dz \\
&= \int_{-1}^{u} \frac{1}{2} \times \frac{1}{3} dz \\
&= \frac{1}{6}[z]_{-1}^{u} = \frac{1}{6}(u+1) \\
&= \frac{(u+1)}{6} \qquad -1 < u < 1
\end{aligned}
$$

Case 2:Width of the window=3-0=3, lower range=1, upper range=-1+3=2 $1 < u < 2$



$$
\begin{aligned}
f_U(u) &= \int_{-\infty}^{\infty} f_Z(z) f_X(u-z)dz \\
&= \int_{-1}^{1} \frac{1}{2} \times \frac{1}{3} dz \\
&= \frac{1}{6}[z]_{-1}^{1} = \frac{1}{6}(1-(-1)) \\
&= \frac{1}{3} \qquad 1 < u < 2
\end{aligned}
$$

Case 3:Width of the window=3-0=3, lower range=2, upper range=1+3=4 $2 < u < 4$



$$
\begin{aligned}
f_U(u) &= \int_{-\infty}^{\infty} f_Z(z) f_X(u-z)dz \\
&= \int_{u-3}^{1} \frac{1}{2} \times \frac{1}{3} dz \\
&= \frac{1}{6}[z]_{u-3}^{1} = \frac{1}{6}(1-(u-3)) \\
&= \frac{4-u}{6} \qquad 2 < u < 4
\end{aligned}
$$

38. Probability density function for two independent random variables X and Y are

$$\begin{aligned} f_X(x) &= ae^{-ax}u(x) \\ f_Y(y) &= (a^3/2)y^2e^{-ay}u(y) \end{aligned}$$

where a=3. If $W = X + Y$ what is $f_W(w)$ [?]

Solution:

$$\begin{aligned} f_W(w) &= \int_{-\infty}^{\infty} f_Y(y)f_X(w-y)dy \\ &= \int_0^w (a^3/2)y^2e^{-ay}ae^{-a(w-y)}dy \\ &= \frac{a^4e^{-aw}}{2}\int_0^w y^2e^{-ay}e^{ay}dy \\ &= \frac{a^4e^{-aw}}{2}\int_0^w y^2dy \\ &= \frac{a^4e^{-aw}}{2}\left[\frac{y^3}{3}\right]_0^w \\ &= \frac{a^4e^{-aw}}{2}\frac{w^3}{3} \\ &= w^3e^{-3w}\frac{3^4}{6} \\ &= 13.5w^3e^{-3w} \end{aligned}$$

39. The pdf for an erlang random variable X of order two is

$$f_X(x) = \lambda^2xe^{-\lambda x} \quad x > 0$$

and is 0 otherwise. The random variable $Y = X_1 + X_2$ where X_1 and X_2 are independent trials of X Find the pdf for Y [?]

Solution:

$$\begin{aligned} f_W(w) &= \int_{-\infty}^{\infty} f_Y(y)f_X(w-y)dy \\ &= \int_0^y \lambda^2xe^{-\lambda x}\lambda^2(y-x)e^{-\lambda(y-x)}dx \\ &= \lambda^4\int_0^y xe^{-\lambda x}(y-x)e^{-\lambda(y-x)}dx \\ &= \lambda^4\int_0^y e^{-\lambda x-\lambda y+\lambda x}[xy-x^2]dx \\ &= \lambda^4e^{-\lambda y}\int_0^y [xy-x^2]dx \\ &= \lambda^4e^{-\lambda y}\left[\frac{x^2}{2}y-\frac{x^3}{3}\right]_0^y \\ &= \lambda^4e^{-\lambda y}\left[\frac{y^2}{2}-\frac{y^3}{3}\right] \\ &= \lambda^4e^{-\lambda y}\left[\frac{3y^3-2y^3}{6}\right] \\ &= \frac{\lambda^4y^3}{6}e^{-\lambda y} \end{aligned}$$

40. Probability density function for two independent random variables Z and V are

$$\begin{aligned} f_Z(z) &= ae^{-az}u(z) \\ f_V(y) &= a^2ve^{-av}u(v) \end{aligned}$$

where$a = \frac{1}{3}$**. If $Y = Z + V$ what is $f_Z(z)$** [?]

Solution:

$$\begin{aligned} f_W(w) &= \int_{-\infty}^{\infty} f_Y(y)f_X(w-y)dy \\ &= \int_0^y a^2ve^{-av}ae^{-a(y-v)}dv \\ &= a^3e^{-ay}\int_0^y vdv \\ &= a^3e^{-ay}\left[\frac{v^2}{2}\right]_0^y \\ &= \frac{a^3}{2}y^2e^{-ay} = 0.0185y^2e^{-ay} \end{aligned}$$

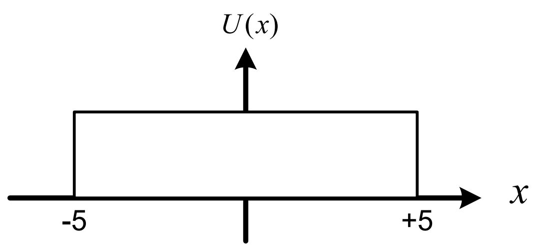
41. Let the random variable U be uniformly distributed between ± 5. Also let the pdf for the random variable V be

$$f_V(v) = 3e^{-3v}u(v)$$

U and V are independent and $W = U + V$. What is the pdf for W [?]

Solution:

The random variable U is uniformly distributed between $\pm 5 = -5\ to\ +5$ it's pdf is



$$f_U(u) = \frac{1}{b-a} = \frac{1}{5-(-5)} = \frac{1}{10}$$

$$\begin{aligned} f_W(w) &= \int_{-\infty}^{\infty} f_U(u)f_V(w-u)du \\ f_W(w) &= 0 \quad w < -5 \\ &= \int_{-5}^{w} \frac{1}{10}3e^{-3(w-u)}du \\ &= \frac{1}{10}3\left[\frac{e^{-3(w-u)}}{3}\right]_{-5}^{w} \\ &= \frac{1}{10}(1 - e^{-3(w+5)} \quad -5 < w < 5 \\ &= \frac{1}{10}\int_{-5}^{5} 3e^{-3(w-u)}du \quad -5 < w < 5 \\ &= \frac{1}{10}[e^{-3(w-5)} - e^{-3(w+5)}] \quad w > 5 \end{aligned}$$

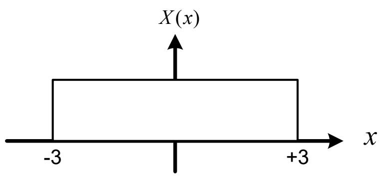
42. It is given that $f_X(x)$ is uniformly distributed between ± 3. Also

$$f_Y(y) = 7e^{-7y}u(y)$$

$W = X + Y$ where X and Y are independent. Find the pdf for W [?]

Solution:

The random variable X is uniformly distributed between $\pm 5 = -3\ to\ +3$ it's pdf is



$$f_U(u) = \frac{1}{b-a} = \frac{1}{3-(-3)} = \frac{1}{6}$$

$$\begin{aligned}
f_W(w) &= \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)dx \\
f_W(w) &= 0 \quad w < -3 \\
&= \int_{-3}^{w} \frac{1}{6} 7e^{-7(w-x)}dx \\
&= \frac{1}{6}7\left[\frac{e^{-7(w-x)}}{7}\right]_{-3}^{w} \\
&= \frac{1}{6}(1-e^{-7(w+3)} \quad -3 < w < 3 \\
&= \frac{1}{6}\int_{-3}^{3} 7e^{-7(w-x)}du \quad -3 < w < 3 \\
&= \frac{1}{6}[e^{-7(w-3)} - e^{-7(w+3)}] \quad w > 3
\end{aligned}$$

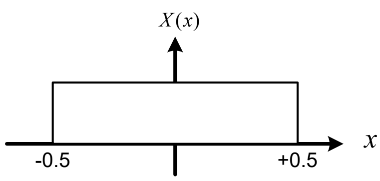
43. The random variable X be uniformly distributed between ± 0.5. The random variable Z has the pdf

$$f_Z(z) = 3e^{-z}u(z)$$

$Y = X + Z$ where X and Z are independent. Find the pdf for Y [?]

Solution:

The random variable X is uniformly distributed between $\pm 5 = -3\ to\ +3$ it's pdf is



$$f_U(u) = \frac{1}{b-a} = \frac{1}{0.5-(-0.5)} = 1$$

$$
\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f_X(x) f_Z(y-x)dx \\
f_Y(y) &= 0 \;\; w < -0.5 \\
&= \int_{-0.5}^{y} 1e^{-(y-x)}dx \\
&= \left[\frac{e^{-(y-x)}}{1}\right]_{-0.5}^{w} \\
&= (1 - e^{-(y+0.5)} \quad -0.5 < w < 0.5 \\
&= \int_{-0.5}^{0.5} e^{-(y-x)}dx \quad -3 < w < 3 \\
&= e^{-(y-0.5)} - e^{-(y+0.5)}] \;\; 0.5 < y
\end{aligned}
$$

44. The random variable X has the pdf $c(7-x)$ for all x between 0 and 7 and is 0 otherwise. The random variable Y is independent of X and is uniformly distributed between 0 and 7. $W = X + Y$. Find the necessary value of c and then find $f_W(w)$ [?]

Solution:

$$
f_Y(y) = \frac{1}{b-a} = \frac{1}{7-0} = \frac{1}{7}
$$

$$
\begin{aligned}
1 &= \frac{1}{2}(7)(7c) \\
c &= = \frac{2}{49}
\end{aligned}
$$

$$
\begin{aligned}
f_W(w) &= \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)dx \\
&= \int_{0}^{w} \frac{2}{49}\frac{1}{7}(7-x)dx \\
&= \frac{2}{343}\int_{0}^{w} (7-x)dx \\
&= \frac{2}{343}\left[7x - \frac{x^2}{2}\right]_0^w \\
&= \frac{2}{343}\left[7w - \frac{w^2}{2}\right] \\
&= \frac{1}{343}(14w - w^2) \;\; 0 < w < 7
\end{aligned}
$$

$$
\begin{aligned}
f_W(w) &= \\
&= \int_{w-7}^{7} \frac{2}{49}\frac{1}{7}(7-x)dx \\
&= \frac{2}{343}\int_{w-7}^{7}(7-x)dx \\
&= \frac{2}{343}\left[7x - \frac{x^2}{2}\right]_{w-7}^{7} \\
&= \frac{2}{343}\left\{\left[7(7) - \frac{(7)^2}{2}\right] - \left[7(w-7) - \frac{(w-7)^2}{2}\right]\right\} \\
&= \frac{w^2 - 28w + 196}{343} \quad 7 < w < 14 \\
&= 0 \; otherwise
\end{aligned}
$$

45. The random variable X has the pdf $c(5-x)$ for all x between 0 and 5 and is 0 otherwise. The random variable Y is independent of X and is uniformly distributed between 0 and 5. $U = X + Y$. Find the necessary value of c and then find $f_U(u)$ [?]

Solution:

$$
f_Y(y) = \frac{1}{b-a} = \frac{1}{5-0} = \frac{1}{5}
$$

$$
\begin{aligned}
1 &= \frac{1}{2}(5)(5c) \\
c &= = \frac{2}{25}
\end{aligned}
$$

$$
\begin{aligned}
f_U(u) &= \int_{-\infty}^{\infty} f_X(x) f_Y(u-x)dx \\
&= \int_0^u \frac{2}{25}\frac{1}{5}(5-x)dx \\
&= \frac{2}{125}\int_0^u (5-x)dx \\
&= \frac{2}{125}\left[5x - \frac{x^2}{2}\right]_0^u \\
&= \frac{2}{125}\left[5u - \frac{u^2}{2}\right] \\
&= \frac{1}{125}(10u - u^2) \quad 0 < u < 5
\end{aligned}
$$

$$
\begin{aligned}
f_U(u) &= \\
&= \int_{u-5}^{5} \frac{2}{25}\frac{1}{5}(5-x)dx \\
&= \frac{2}{125}\int_{u-5}^{5}(5-x)dx \\
&= \frac{2}{125}\left[5x-\frac{x^2}{2}\right]_{w-5}^{5} \\
&= \frac{2}{125}\left\{\left[5(5)-\frac{(5)^2}{2}\right]-\left[5(w-5)-\frac{(w-5)^2}{2}\right]\right\} \\
&= \frac{u^2-20u+100}{125} \quad 5<w<10 \\
&= 0 \; otherwise
\end{aligned}
$$

46. The random variable X has the pdf $c(3-x)$ for all x between 0 and 3 and is 0 otherwise.

The random variable Y is independent of X and is uniformly distributed between 0 and 3. $V = X + Y$. Find the necessary value of c and then find $f_V(v)$ [?]

Solution:

$$
f_Y(y) = \frac{1}{b-a} = \frac{1}{3-0} = \frac{1}{3}
$$

$$
\begin{aligned}
1 &= \frac{1}{2}(3)(3c) \\
c &= = \frac{2}{9}
\end{aligned}
$$

$$
\begin{aligned}
f_V(v) &= \int_{-\infty}^{\infty} f_X(x)f_Y(v-x)dx \\
&= \int_0^v \frac{2}{9}\frac{1}{3}(3-x)dx \\
&= \frac{2}{27}\int_0^v(3-x)dx \\
&= \frac{2}{27}\left[3x-\frac{x^2}{2}\right]_0^v \\
&= \frac{2}{27}\left[3v-\frac{v^2}{2}\right] \\
&= \frac{1}{27}(6v-v^2) \quad 0<v<3
\end{aligned}
$$

$$
\begin{aligned}
f_U(u) &= \\
&= \int_{v-3}^{3} \frac{2}{9}\frac{1}{3}(3-x)dx \\
&= \frac{2}{27}\int_{v-3}^{3}(3-x)dx \\
&= \frac{2}{27}\left[3x - \frac{x^2}{2}\right]_{v-3}^{3} \\
&= \frac{2}{27}\left\{\left[3(3) - \frac{(3)^2}{2}\right] - \left[3(v-3) - \frac{(v-3)^2}{2}\right]\right\} \\
&= \frac{v^2 - 12v + 36}{27} \quad 3 < v < 6 \\
&= 0 \; otherwise
\end{aligned}
$$

47. A discrete random variable Y has the pdf

$$f_Y(y) = 0.5\delta(y) + 0.5\delta(y-3)$$

$U = Y_1 + Y_2$ **where $Y's$ are independent. What is the pdf for U ?** [?]

Solution:

$$
\begin{aligned}
f_U(u) &= \int_{-\infty}^{\infty} f_Y(y) f_Y(u-y)dy \\
&= \int_{-\infty}^{\infty} [0.5\delta(y) + 0.5\delta(y-3)][0.5\delta(u-y) + 0.5\delta(u-y-3)]dy \\
&= \int_{-\infty}^{\infty} [0.25\delta(y)\delta(u-y) + 0.25\delta(y-3)\delta(u-y) + 0.25\delta(y)\delta(u-y-3) + 0.25\delta(y-3)\delta(u-y-3)]dy \\
&= 0.25\delta(u) + 0.5\delta(u-3) + 0.25\delta(u-6)
\end{aligned}
$$

48. A discrete random variable Z has the pdf

$$f_Z(z) = 0.3\delta(z-1) + 0.7\delta(z-2)$$

$V = Z_1 + Z_2$ **where $Z's$ are independent. What is the pdf for V ?** [?]

Solution:

$$
\begin{aligned}
f_V(v) &= \int_{-\infty}^{\infty} f_Z(z) f_Z(v-z)dz \\
&= \int_{-\infty}^{\infty} [0.3\delta(z-1) + 0.7\delta(z-2)][0.3\delta(v-z-1) + 0.7\delta(v-z-2)]dz \\
&= \int_{-\infty}^{\infty} [0.09\delta(z-1)\delta(v-z-1) + 0.21\delta(z-2)\delta(v-z-1) + 0.21\delta(z-1)\delta(v-z-1) + 0.49\delta(z-2)\delta(v-z-2)]dz \\
&= 0.09\delta(v-2) + 0.42\delta(v-3) + 0.49\delta(v-4)
\end{aligned}
$$

49. A discrete random variable Y has the pdf

$$f_X(x) = 0.6\delta(x-2) + 0.4\delta(x-1)$$

$W = X_1 + X_2$ **where $X's$ are independent. What is the pdf for W ?** [?]

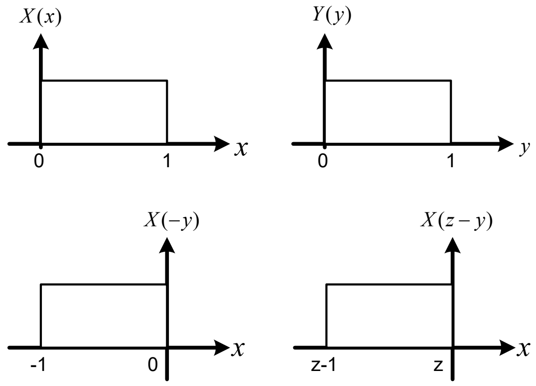
Solution:

$$
\begin{aligned}
f_W(w) &= \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)dx \\
&= \int_{-\infty}^{\infty} [0.6\delta(x-2) + 0.4\delta(x-1)][0.6\delta(x-2) + 0.4\delta(x-1))]dx \\
&= \int_{-\infty}^{\infty} [0.16\delta(x-1)\delta(w-x-1) + 0.24\delta(x-2)\delta(w-x-1) + 0.24\delta(x-1)\delta(w-x-2) + 0.36\delta(x-2)\delta(w-x-2)]dx \\
&= 0.16\delta(w-2) + 0.48\delta(w-3) + 0.36\delta(w-4)
\end{aligned}
$$

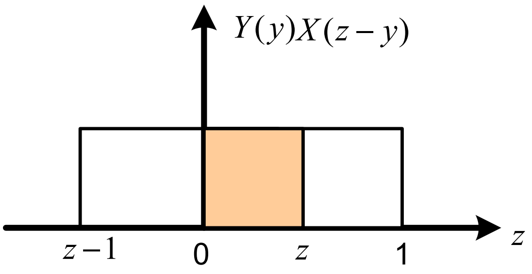
35. Let X and Y be independent uniform random variables over (0, 1). Find and sketch the pdf of $Z = X + Y$. [?]

Solution:

$$\begin{aligned} f_X(x) &= \frac{1}{b-a} = \frac{1}{1-(0)} = 1 \\ f_Y(y) &= \frac{1}{b-a} = \frac{1}{1-(0)} = 1 \end{aligned}$$

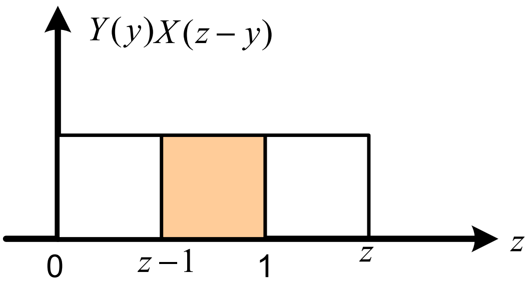


Case 1: $0 < z < 1$



$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_Y(y) f_X(z-y) dy \\ &= \int_0^z (1) \times (1) dy \\ &= [y]_0^z = [z-0] \\ &= z \qquad 0 < z < 1 \end{aligned}$$

Case 2: $0 < (z-1 < 1) \rightarrow 1 < z < 2$



$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(x) f_X(y-x) dy \\ &= \int_{z-1}^{1} (1) \times (1) dy \\ &= [y]_{z-1}^{1} = [1-(z-1)] \\ &= 2-z \qquad 1 < z < 2 \end{aligned}$$

The sketch of pdf $Z = X + Y$

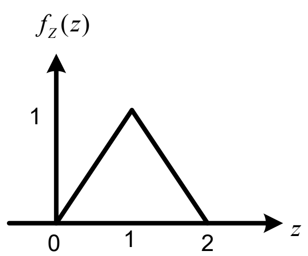


Figure 1.5: sketch of pdf $Z = X + Y$

Note: Entire material is taken from different text books or from the Internet (different websites). Slightly it is modified from the original content. It is not for any commercial purpose. It is used to teach students. Suggestions are always encouraged.

1.5 Sums of IID Random Variables

Consider a situation when each random variable is added its sum is having the same pdf. The pdf associated with each $f_(x)$. This is denoted as independent and identically distributed (IID) random variables

$$W = \sum_{i=1}^{n} X_i$$

Then the mean and variance of X_i is

$$E[X_i] = E[X] = \mu_X$$

The variance of X_i is

$$Var[X_i] = Var[X] = \sigma_X^2$$

$$E[X_iX_j] = \begin{cases} E[X_i^2] = E[X^2] = \mu_X^2 + \sigma_X^2 & j = i \\ E[X_iX_i] = \mu_X^2 & j \neq i \end{cases}$$

When $n = 2$

$$W_2 = X_1 + X_2$$

$$E[W_2] = 2\mu_X$$

The variance of W_2 is

$$Var[W_2] = 2\sigma_X^2$$

When $n = 3$

$$\begin{aligned} W_3 &= X_1 + X_2 + X_3 \\ &= W_2 + X_3 \end{aligned}$$

$$E[W_3] = 3\mu_X$$

The variance of W_3 is

$$Var[W_3] = 3\sigma_X^2$$

Similarly continued then

$$W = W_{n-1} + X_n$$

$$\mu_W = n\mu_X$$

The variance of W_3 is

$$\sigma^2 W = n\sigma_X^2$$

53. The random variable U has a mean of 0.3 and a variance of 1.5

a) Find the mean and variance of Y if

$$Y = \frac{1}{53}\sum_{i=1}^{53} U_i$$

b) Find the mean and variance of Z if

$$Z = \sum_{i=1}^{53} U_i$$

In these two sums, the $U_i's$ are IID [?]

Solution:

a) The mean and variance of Y is
$\mu_U = 0.3,\ \sigma_U^2 = 1.5$

$$\begin{aligned} \mu_Y &= \mu_U = 0.3 \\ \sigma_Y^2 &= \frac{\sigma_U^2}{n} = \frac{1.5}{53} = 0.0283 \end{aligned}$$

b) The mean and variance of Z is

$$\begin{aligned} \mu_Z &= n\mu_U = 53(0.3) = 15.9 \\ \sigma_Z^2 &= n\sigma_U^2 = 53(1.5) = 79.5 \end{aligned}$$

54. The random variable X is uniformly distributed between ± 1

a) Find the mean and variance of Y if

$$Y = \frac{1}{37}\sum_{i=1}^{37} X_i$$

b) Find the mean and variance of Z if

$$Z = \sum_{i=1}^{37} X_i$$

In these two sums, the $X_i's$ are IID [?]

Solution:

a) The mean and variance of Y is
$\mu_X = 0,\ \sigma_U^2 = \frac{2^2}{12} = 0.333$

$$\begin{aligned} \mu_Y &= \mu_X = 0 \\ \sigma_Y^2 &= \frac{\sigma_X^2}{n} = \frac{0.3333}{37} = 0.009 \end{aligned}$$

b) The mean and variance of Z is

$$\begin{aligned} \mu_Z &= n\mu_X = 0 \\ \sigma_Z^2 &= n\sigma_X^2 = 37(0.3333) = 12.3333 \end{aligned}$$

55. The random variable V has a mean of 1 and a variance of 4

a) Find the mean and variance of Y if

$$Y = \frac{1}{87}\sum_{i=1}^{87} V_i$$

b) Find the mean and variance of Z if

$$Z = \sum_{i=1}^{87} V_i$$

In these two sums, the $V_i's$ are IID [?]

Solution:

a) The mean and variance of Y is
$\mu_V = 1,\ \sigma_V^2 = 4$

$$\begin{aligned}\mu_Y &= \mu_V = 1 \\ \sigma_Y^2 &= \frac{\sigma_V^2}{n} = \frac{4}{87} = 0.0460\end{aligned}$$

b) The mean and variance of Z is

$$\begin{aligned}\mu_Z &= n\mu_V = 87(1) = 87 \\ \sigma_Z^2 &= n\sigma_V^2 = 87(4) = 348\end{aligned}$$

56. The random variable X has a mean of 12.6 and a variance of 2.1. The random variable Y is related to X by $Y = 10(X - \mu_X)$. The random variable Z is as shown here.

$$Z = \sum_{i=1}^{100} Y_i$$

where $Y_i's$ are IID. What are μ_Z and σ_Z^2 [?]

Solution:

$\mu_X = 12.6,\ \sigma_X^2 = 2.1$

$$\begin{aligned}\mu_Y &= 10(\mu_X - \mu_X) = 0 \\ \sigma_Y^2 &= 10^2\sigma_Y^2 = 210 \\ \mu_Z &= 100\mu_Y = 0 \\ \sigma_Y^2 &= 100\sigma_Y^2 = 21000\end{aligned}$$

57. The random variable $X = 3 + V$, where V is a Gaussian random variable with a mean of 0 and a variance of 30. Seventy two independent realizations of X are averaged.

$$Y = \frac{1}{72}\sum_{i=1}^{72} X_i$$

What are mean and variance of Y [?]

Solution:

$\mu_V = 0,\ \sigma_V^2 = 30$

$$\begin{aligned}
\mu_X &= 3 + \mu_V = 3 \\
\sigma_X^2 &= 1^2\sigma_V^2 = 30 \\
\mu_Y &= 0 \\
\sigma_Y^2 &= \frac{\sigma_X^2}{72} = \frac{30}{72} = 0.4167
\end{aligned}$$

58. X is random variable with a variance of 1.8 and a mean of 14 and. $Y = X - \mu_X$. Z is as shown here.

$$Z = \frac{1}{100}\sum_{i=1}^{100} Y_i$$

where $Y_i's$ are IID. What are mean and variance of Z [?]

Solution:

$\mu_X = 14,\ \sigma_X^2 = 1.8$

$$\begin{aligned}
\mu_Y &= \mu_X - \mu_X = 0 \\
\sigma_Y^2 &= 1^2\sigma_X^2 = 1.8 \\
\mu_Z &= \mu_Y = 0 \\
\sigma_Z^2 &= \frac{\sigma_Y^2}{100} = 0.0180
\end{aligned}$$

59. The random variable Z is uniformly distributed between 0 and 1. The random variable Y is obtained from Z as follows

$$Y = 3Z + 5.5$$

One hundred independent realizations of Y are averaged

$$U = \frac{1}{100}\sum_{i=1}^{100} Y_i$$

a) **Estimate the probability $P(U \le 7.1)$**

b) **If 1000 independent calculations of U are performed, approximately how many of these calculated values for U would be less than 7.1?**

[?]

Solution:

$$\begin{aligned}
\mu_Z &= \frac{0+1}{2} = 0.5 \\
\sigma_Z^2 &= \frac{b-a}{12} = \frac{1-0}{12} = \frac{1}{12} \\
\mu_Y &= 3\mu_Z + 5.5 = 3(0.5) + 5.5 = 7 \\
\sigma_Y^2 &= 3^2\sigma_Z^2 = \frac{9}{12} \\
\mu_U &= \mu_Y = 7 \\
\sigma_U &= \sqrt{\frac{9}{1200}} = 0.0866
\end{aligned}$$

a) The probability $P(U \leq 7.1)$

$$\begin{aligned} P(U \leq 7.1) &= F_U(7.1) = \phi\left(\frac{x-\mu}{\sigma}\right) \\ &= \phi\left(\frac{7.1-7}{0.0866}\right) \\ &= \phi(1.1547) \quad \textit{From Z table} \\ \sigma_Y^2 &= = 0.8759 \end{aligned}$$

b)

$$P(U \leq 7.1) \times 1000 = 876$$

60. The random variable Z is uniformly distributed between 0 and 1. The random variable Y is obtained from Z as follows

$$Y = 3.5Z + 5.25$$

One hundred independent realizations of Y are averaged

$$V = \frac{1}{100}\sum_{i=1}^{100} Y_i$$

a) Estimate the probability $P(V \leq 7.1)$

b) If 1000 independent calculations of V are performed, approximately how many of these calculated values for V would be less than 7.1?

[?]

Solution:

$$\begin{aligned} \mu_Z &= \frac{0+1}{2} = 0.5 \\ \sigma_Z^2 &= \frac{b-a}{12} = \frac{1-0}{12} = \frac{1}{12} \\ \mu_Y &= 3.5\mu_Z + 5.25 = 3.5(0.5) + 5.25 = 7 \\ \sigma_Y^2 &= (3.5)^2\sigma_Z^2 = \frac{(3.5)^2}{12} \\ \mu_U &= \mu_Y = 7 \\ \sigma_U &= 3.5\sqrt{\frac{1}{1200}} = 0.1010 \end{aligned}$$

a) The probability $P(U \leq 7.1)$

$$\begin{aligned} P(U \leq 7.1) &= F_U(7.1) = \phi\left(\frac{x-\mu}{\sigma}\right) \\ &= \phi\left(\frac{7.1-7}{0.1010}\right) \\ &= \phi(0.9900) \quad \textit{From Z table} \\ \sigma_Y^2 &= = 0.8389 \end{aligned}$$

b)

$$P(U \leq 7.1) \times 1000 = 839$$

61. The random variable Z is uniformly distributed between 0 and 1. The random variable Y is obtained from Z as follows

$$Y = 2.5Z + 5.75$$

One hundred independent realizations of Y are averaged

$$W = \frac{1}{100}\sum_{i=1}^{100} Y_i$$

a) **Estimate the probability** $P(W \le 7.1)$

b) **If 1000 independent calculations of W are performed, approximately how many of these calculated values for W would be less than 7.1?**

[?]

Solution:

$$\begin{aligned}
\mu_Z &= \frac{0+1}{2} = 0.5 \\
\sigma_Z^2 &= \frac{b-a}{12} = \frac{1-0}{12} = \frac{1}{12} \\
\mu_Y &= 2.5\mu_Z + 5.75 = 2.5(0.5) + 5.75 = 7 \\
\sigma_Y^2 &= (2.5)^2\sigma_Z^2 = \frac{(2.5)^2}{12} \\
\mu_U &= \mu_Y = 7 \\
\sigma_U &= 2.5\sqrt{\frac{1}{1200}} = 0.0722
\end{aligned}$$

a) **The probability** $P(U \le 7.1)$

$$\begin{aligned}
P(U \le 7.1) &= F_U(7.1) = \phi\left(\frac{x-\mu}{\sigma}\right) \\
&= \phi\left(\frac{7.1-7}{0.0722}\right) \\
&= \phi(1.3850) \quad \textit{From Z table} \\
\sigma_Y^2 &= = 0.9170
\end{aligned}$$

b)

$$P(U \le 7.1) \times 1000 = 917$$

Note: Entire material is taken from different text books or from the Internet (different websites). Slightly it is modified from the original content. It is not for any commercial purpose. It is used to teach students. Suggestions are always encouraged.

1.6 Conditional Joint Probabilities

The conditioned cdf of a bivariate random variable is defined as

$$F_{XY}(x,y|B) = \frac{P\{(X \le x) \cap (Y \le y) \cap B\}}{P(B)}$$

The joint pdf conditioned by an event B is defined as

$$f_{XY}(x,y|B) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y|B)$$

The event B is a set of bivariate observations (x,y) in the (x) (y) plane

$$P(B) = \int\int_B f_{XY}(x,y)dxdy$$

The above equation is convolution hence it can be written as

$$f_{XY}(x,y|B) = \begin{cases} \frac{f_{XY}(x,y)}{P(B)} & (x,y) \in B \\ 0 & otherwise \end{cases}$$

Conditional joint pdf for y is

$$f_Y(y|B) = \int\int_B f_{XY}(x,y|B)dx$$

$$\begin{aligned} f_Y(y|B) &= \int_x\int_{x+dx} f_{XY}(u,y|B)du \\ &= \int_x\int_{x+dx} \frac{f_{XY}(u,y)}{P(B)}du \\ &= \frac{f_{XY}(x,y)}{P(B)}dx \end{aligned}$$

$$\begin{aligned} f_Y(y|X=x) &= \frac{f_{XY}(x,y)}{f_X(x)} \\ f_Y(y|x) &= \frac{f_{XY}(x,y)}{f_X(x)} \end{aligned}$$

Similarly

$$f_X(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

Conditional cdfs are

$$\begin{aligned} F_Y(y|x) &= \int_{-\infty}^{y} f_Y(u|x)du \\ F_X(x|y) &= int_{-\infty}^{x} f_X(u|y)du \end{aligned}$$

Conditional expectations are

$$\begin{aligned} E[g_1(Y)|x] &= \int_{-\infty}^{\infty} g_1(y)f_Y(y|x)dy \\ E[g_2(X)|y] &= \int_{-\infty}^{\infty} g_2(x)f_X(x|y)dx \end{aligned}$$

Conditional mean and variance are

$$\begin{aligned}\mu_{Y|x} &= \int_{-\infty}^{\infty} y f_Y(y|x)dy \\ \sigma^2_{Y|x} &= \int_{-\infty}^{\infty} (y-\mu_{Y|x})^2 f_Y(y|x)dy\end{aligned}$$

$$\begin{aligned}\mu_{X|y} &= \int_{-\infty}^{\infty} x f_X(x|y)dx \\ \sigma^2_{X|y} &= \int_{-\infty}^{\infty} (X-\mu_{X|y})^2 f_X(x|y)dx\end{aligned}$$

The detailed solutions are given in Exercise 11. Refer previous results.

62. Refer to Figure 3.20 used in Exercise 11. Find using (3.113), the pdf of Y conditioned by $X = 1$. Then verify that the conditional pdf satisfies (2.12). Finally, find the mean and the variance of Y conditioned by $X = 1$. [?]

Solution:

It is given that

$$\begin{aligned} f_{XY}(x,y) &= \frac{1}{8} \\ f_X(x) &= \frac{1}{8}(2-x) \end{aligned}$$

$$\begin{aligned} f_Y(y|x) &= \frac{f_{XY}(x,y)}{f_X(x)} \\ &= \frac{\frac{1}{8}}{\frac{1}{8}(2-x)} \quad -2 < x < 2 \end{aligned}$$

When $X = 1$

$$f_Y(y|1) = \frac{1}{(2-x)} = \frac{1}{(2-1)} = 1 \qquad 1 < y < 2, when\ x = 1$$

$$\begin{aligned} \int_{-\infty}^{\infty} f_Y(y|1)dy &= \int_1^2 1dy = [y]_1^2 = [2-1] \\ &= 1 \end{aligned}$$

Conditional mean and variance are

$$\begin{aligned} \mu_{Y|x=1} &= \int_{-\infty}^{\infty} y f_Y(y|x)dy \\ &= \int_1^2 ydy = \left[\frac{y^2}{2}\right]_1^2 \\ &= \frac{1}{2}[4-1] = \frac{3}{2} \end{aligned}$$

$$\begin{aligned} \sigma^2_{Y|x} &= \int_{-\infty}^{\infty} (y - \mu_{Y|x})^2 f_Y(y|x)dy \\ &= \overline{y^2} - (\mu_{Y|x})^2 \end{aligned}$$

$$\begin{aligned} \overline{y^2} &= \int_1^2 y^2 dy = \left[\frac{y^3}{3}\right]_1^2 \\ &= \frac{1}{3}[8-1] = \frac{7}{3} \\ \sigma^2_{Y|x=1} &= \frac{7}{3} - \left(\frac{3}{2}\right)^2 = \frac{28-27}{12} \\ &= \frac{1}{12} \end{aligned}$$

The detailed solutions are given in Exercise 12. Refer previous results.

63. Refer to Figure 3.21 used in Exercise 12. Find using (3.113), the pdf of Y conditioned by $X = 1$. Then verify that the conditional pdf satisfies (2.12). Finally, find the mean and the variance of Y conditioned by $X = 1$. [?]

Solution:

It is given that

$$\begin{aligned} f_{XY}(x,y) &= \frac{1}{8} \\ f_X(x) &= \frac{1}{8}(2+x) \end{aligned}$$

$$\begin{aligned} f_Y(y|x) &= \frac{f_{XY}(x,y)}{f_X(x)} \\ &= \frac{\frac{1}{8}}{\frac{1}{8}(2+x)} \quad -2 < x < 2 \end{aligned}$$

When $X = 1$

$$f_Y(y|1) = \frac{1}{(2+x)} = \frac{1}{(2+1)} = \frac{1}{3} \quad -1 < y < 2, when\ x = 1$$

$$\begin{aligned} \int_{-\infty}^{\infty} f_Y(y|1)dy &= \int_{-1}^{2} \frac{1}{3}dy = \frac{1}{3}[y]_{-1}^{2} = \frac{1}{3}[2+1] \\ &= 1 \end{aligned}$$

Conditional mean and variance are

$$\begin{aligned} \mu_{Y|x=1} &= \int_{-\infty}^{\infty} y f_Y(y|x)dy \\ &= \int_{-1}^{2} \frac{1}{3}ydy = \frac{1}{3}\left[\frac{y^2}{2}\right]_{-1}^{2} \\ &= \frac{1}{3}\frac{1}{2}[4-1] = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \sigma^2_{Y|x} &= \int_{-\infty}^{\infty} (y-\mu_{Y|x})^2 f_Y(y|x)dy \\ &= \overline{y^2} - (\mu_{Y|x})^2 \end{aligned}$$

$$\begin{aligned} \overline{y^2} &= \int_{-1}^{2} y^2dy = \left[\frac{y^3}{3}\right]_{2}^{-1} \\ &= \frac{1}{3}[8+1] = \frac{7}{3} \\ \sigma^2_{Y|x=1} &= \frac{7}{3} - \left(\frac{1}{2}\right)^2 = \frac{28-27}{12} \\ &= \frac{1}{12} \end{aligned}$$

64. Refer to Figure 3.22 used in Exercise 13. Find using (3.113), the pdf of Y conditioned by $X = 1$. Then verify that the conditional pdf satisfies (2.12). Finally, find the mean and the variance of Y conditioned by $X = 1$. [?]

Solution:

It is given that

$$\begin{aligned} f_{XY}(x,y) &= \frac{1}{8} \\ f_X(x) &= \frac{1}{8}(x+2) \end{aligned}$$

$$\begin{aligned} f_Y(y|x) &= \frac{f_{XY}(x,y)}{f_X(x)} \\ &= \frac{\frac{1}{8}}{\frac{1}{8}(x+2)} \quad -2 < x < 2 \end{aligned}$$

When $X = 1$

$$f_Y(y|1) = \frac{1}{(x+2)} = \frac{1}{(1+2)} = \frac{1}{3} \quad -2 < y < 1, when\ x = 1$$

$$\begin{aligned} \int_{-\infty}^{\infty} f_Y(y|1)dy &= \int_{-2}^{1} \frac{1}{3}dy = \frac{1}{3}[y]_{-2}^{1} = \frac{1}{3}[1+2] \\ &= 1 \end{aligned}$$

Conditional mean and variance are

$$\begin{aligned} \mu_{Y|x=1} &= \int_{-\infty}^{\infty} y f_Y(y|x)dy \\ &= \int_{-2}^{1} \frac{1}{3} y dy = \frac{1}{3}\left[\frac{y^2}{2}\right]_{-2}^{1} \\ &= \frac{1}{3}\frac{1}{2}[1-4] = -\frac{1}{2} \end{aligned}$$

$$\begin{aligned} \sigma^2_{Y|x=1} &= \int_{-\infty}^{\infty} (y - \mu_{Y|x})^2 f_Y(y|x)dy \\ &= \overline{y^2} - (\mu_{Y|x})^2 \end{aligned}$$

$$\begin{aligned} \overline{y^2} &= \int_{-2}^{1} y^2 dy = \left[\frac{y^3}{3}\right]_{-2}^{1} \\ &= \frac{1}{3}[1+8] = \frac{7}{3} \\ \sigma^2_{Y|x} &= \frac{7}{3} - \left(\frac{1}{2}\right)^2 = \frac{28-27}{12} \\ &= \frac{1}{12} \end{aligned}$$

65. Refer to the joint pdf $f_{XY}(x,y)$ given in Exercise 9. Find using (3.113), the pdf of Y conditioned by $X=2$. Then verify that the conditional pdf satisfies (2.12). Finally, find the mean and the variance of Y conditioned by $X=2$. [?]

Solution:

It is given that

$$\begin{aligned} f_{XY}(x,y) &= \frac{1}{8} \\ f_X(x) &= \frac{1}{8}(x+2) \end{aligned}$$

$$\begin{aligned} f_Y(y|x) &= \frac{f_{XY}(x,y)}{f_X(x)} \\ &= \frac{\frac{1}{8}}{\frac{1}{8}(x+2)} \quad -2 < x < 2 \end{aligned}$$

When $X=2$

$$f_Y(y|2) = \frac{\sqrt{2\pi}}{1.9079\pi} exp\left[\frac{-(4-1.2y+y^2)}{1.82}+\frac{4}{2}\right]$$

$$f_Y(y|2) = \frac{1}{\sqrt{2\pi 0.91}} exp\left[\frac{-(y-0.6)^2}{2(0.91)}\right]$$

Note: Entire material is taken from different text books or from the Internet (different websites). Slightly it is modified from the original content. It is not for any commercial purpose. It is used to teach students. Suggestions are always encouraged.

1.7 Selected Topics

1.7.1 Chi Square Random Variables

The random variable V where, for integers for $r \geq 1$

$$V = \sum_{i=1^r} Z_i^2$$

The random variable Z is the normalized Gaussian random variable defined as

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{Z^2}{2}} \qquad -\infty < z < \infty$$

The event expectations for the random variable Z

$$\begin{aligned} E[Z] &= \mu_Z = 0 \\ E[Z^2] &= \sigma_Z^2 = 1 \end{aligned}$$

Consider a new random variable Y which defined as

$$Y = Z^2$$

Conditional joint pdf for Y is

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi y}} e^{-\frac{y2}{2}} & y > 0 \\ 0 & y < 0 \end{cases}$$

The expectations of Y are

$$\begin{aligned} E[Z^2] &= \mu_Y = 1 \\ E[Y^2] &= E[Z^2] = 3 \\ \sigma_Y^2 &= E[Y^2] - \mu_Y^2 = 2 \end{aligned}$$

The Characteristic function of Y is

$$\phi_{j\omega} = (1 - j2\omega)^{-\frac{1}{2}}$$

When r=1 then $V_1 = Y_1 = Y$ and

$$\begin{aligned} f_{V_1}(v) &= f_Y(v) \\ E_[V_1] &= \mu_Y = 1 \\ Var[V_1] &= \sigma_Y^2 = 2 \\ \phi_{V_1}(j\omega) &= (1 - j2\omega)^{-\frac{1}{2}} \end{aligned}$$

When r=2 then $V_2 = Z_1 + Z_2 = Y_1 + Y_2 = V_1 + Y$

$$\begin{aligned} E_[V_2] &= 2\mu_Y = 2 \\ Var[V_2] &= 2\sigma_Y^2 = 4 \\ \phi_{V_2}(j\omega) &= \phi_{V_1}(j\omega)^2 = (1 - j2\omega)^{-1} \end{aligned}$$

Conditional joint pdf for Y is

$$f_{V_2}(v) = \begin{cases} \frac{1}{2} e^{-\frac{v2}{2}} & v > 0 \\ 0 & v < 0 \end{cases}$$

When r=3 then $V_3 = Z_1 + Z_2 + Z_3 = Y_1 + Y_2 + Y_3 = V_2 + Y$

$$\begin{aligned} E[V_3] &= E[V_2] + \mu_Y = 3 \\ Var[V_3] &= Var[V_2] + \sigma_Y^2 = 6 \\ \phi_{V_3}(j\omega) &= [\phi_Y(j\omega)]^3 = (1 - j2\omega)^{-\frac{3}{2}} \end{aligned}$$

$$f_{V_3}(v) = \begin{cases} \sqrt{\frac{v}{2}\pi}e^{-\frac{v2}{2}} & v > 0 \\ 0 & v < 0 \end{cases}$$

Continuing and in general $r \geq 1$

$$f_V(v) = \begin{cases} \frac{1}{\tau(r/2)2^{r/2}}v^{(}\frac{r}{2} - 1)e^{-\frac{v2}{2}} & v > 0 \\ 0 & v < 0 \end{cases}$$

1.7.2 Student's t Random Variables

The random variable T where, for integers for $r \geq 1$

$$T = \frac{Z}{\sqrt{V/r}}$$

The joint pdf

$$f_{TV} = \int_0^\infty f_{TV}(t,v)dv$$

By exchanging (t,v) with (x,y)

$$f_{TV} = \int_0^\infty f_T(t|v)f_V(v)dv$$

$$T = \sqrt{\frac{r}{v}}Z$$

$$f_T(t|v) = \sqrt{\frac{v}{r}}f_Z(\sqrt{v/rt})$$

$$f_T(t|v) = \sqrt{\frac{v}{2\pi r}}e^{-(r^2/r)(v/2)}$$

$$f_T(t) = \frac{1}{\sqrt{2\pi r}(r/2)(2^{r/2})}\int_0^\infty v^{[(r+1)/2-l]}e^{-(1+t^2/r)(v/2)}dv$$

Let

$$w = (1+t^2/r)(v/2)$$

$$f_T(t) = \frac{1}{\sqrt{2\pi r}\tau(r/2)(1+t^2/r)^(r+1/2)}\int_0^\infty w^{[(r+1)/2-l]}e^{-w}dw$$

$$f_T(t) = \frac{\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{\pi r}\Gamma(r/2)(1+t^2/r)^(r+1/2)}$$

1.7.3 Cauchy Random Variables

Consider a random variable X which is zero mean Gaussian random variable and another variable Y which is zero mean Gaussian and these two related by the following relation

$$W = a\frac{X}{Y}$$

$$f_X(x) = a\frac{1}{\sigma\sqrt{2\pi}}e^{-x^2/2} \qquad -\infty < x < \infty$$

$$f_Y(y) = a\frac{1}{\sigma\sqrt{2\pi}}e^{-y^2/2} \qquad -\infty < y < \infty$$

Assume that the joint pdf $f_{WY}(w,y)$ is known then the pdf for the random variable W is

$$f_W(w) = \int -\infty^{\infty} f_{WY}(w,y)dy$$

$$f_W(w) = \int -\infty^{\infty} f_W(w|y)f_Y(y)dy$$

If y is variable with range $-\infty < y < \infty$ then

$$W = \frac{a}{y}X$$

$$f_W(w|y) = (y/a)f_X(wy|a)|y \geq 0 - (y/a)f_X(wy/a)|y \leq 0$$

$$f_W(w) = \int -\infty^{\infty}(y/a)f_X(wy|a)f_Y(y)dy + \int -\infty^{\infty}(y/a)f_X(wy/a)f_Y(y)dy$$

$$f_W(w) = \frac{1}{a\pi\sigma^2}\int_0^{\infty} exp\left[-(1+(w/a)^2y^2/2\sigma^2\right] ydy$$

$v = (1+(w/a)^2y^2/2\sigma^2$

$$f_W(w) = \frac{a}{\pi(w^2+a^2)} \qquad -\infty < y < \infty \quad a > 0$$

The cdf is

$$\begin{aligned} F_W(w) &= \int_{-\infty}^{w} f_W(x) \\ &= \frac{1}{\pi}tan^{-1}\left(\frac{w}{a}\right) + \frac{1}{2} \qquad -\infty < x < \infty \quad a > 0 \end{aligned}$$

The characteristic function is

$$\phi_W(jw) = exp(-a|w|) \qquad -\infty < x < \infty$$

1.7.4 Rayleigh Random Variables [?]

Consider a two independent Gaussian random variable X and Y with zero mean and same variance σ and are expressed in the following relation

$$\begin{aligned} f_X(x) &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} \\ f_y(y) &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y}{\sigma}\right)^2} \end{aligned}$$

$$\begin{aligned} f_{XY}(x,y) &= f_X(x) \times f_y(y) \\ &= \frac{1}{\sigma^2 2\pi} e^{-\frac{1}{2\sigma^2}\left(x^2+y^2\right)} \end{aligned}$$

Let

$$x = rcos\theta \quad y = rsin\theta \quad 0 \le r < \infty \qquad 0 \le \theta \le 2\pi$$

$$r = \sqrt{x^2+y^2}$$

$$dxdy = rdrd\theta$$

$$\begin{aligned} f_{XY}(x,y)dxdy &= P(r,\theta)drd\theta \\ P(r,\theta)drd\theta &= \frac{r}{\sigma^2 2\pi} e^{-\frac{1}{2\sigma^2}\left(r^2\right)} \end{aligned}$$

$$\begin{aligned} P(r)\theta &= \int_0^{2\pi} P(r,\theta) \\ &= \int_0^{2\pi} \frac{r}{\sigma^2 2\pi} e^{-\frac{1}{2\sigma^2}\left(r^2\right)} d\theta \\ &= \frac{r}{\sigma^2 2\pi} e^{-\frac{1}{2\sigma^2}\left(r^2\right)} \left[\theta\right]_0^{2\pi} \\ &= \frac{r}{\sigma^2} e^{-\frac{1}{2\sigma^2}\left(r^2\right)} \end{aligned}$$

$$f(r) = \begin{cases} \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} & r \ge 0 \\ 0 & Otherwise \end{cases}$$

$$f(r) = \frac{2r}{b} e^{-\frac{r^2}{b}} \quad r \ge 0$$

$$f_R(r) = \begin{cases} \frac{2r}{b} e^{-\frac{r^2}{b}} & r \ge 0 \\ \quad r \ge 0 & \\ 0 & Otherwise \end{cases}$$

$$F_R(r) = \begin{cases} 1 - e^{-\frac{r^2}{b}} & r \ge 0 \\ \quad r \ge 0 & \\ 0 & Otherwise \end{cases}$$

1.7.5 Central Limit Theorem

Central Limit Theorem states that the sums of independent and identically distributed (IID) random variables can become a Gaussian random variable.

Let X_1, X_2, X_3, X_n are independent and identically distributed (IID) random variables, then their sum is

$$W = \sum_{i=1}^{n} X_i$$

For independent random variables X and Y, the distribution f_Z of $Z = X + Y$equals the convolution of f_X and f_Y:

$$\begin{aligned} f_X(x) &= \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}} \\ f_Y(y) &= \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{(x-\mu_Y)^2}{2\sigma_Y^2}} \end{aligned}$$

By taking Fourier transform

$$\begin{aligned} \mathcal{F}\{f_X\} &= \mathcal{F}_X(\omega) = exp[-j\omega\mu_X]exp\left[-\frac{\sigma_X^2\omega^2}{2}\right] \\ \mathcal{F}\{f_Y\} &= \mathcal{F}_Y(\omega) = exp[-j\omega\mu_Y]exp\left[-\frac{\sigma_Y^2\omega^2}{2}\right] \end{aligned}$$

$$\begin{aligned} f_Z(z) &= (f_X * f_Y)(z) \\ &= F^{-1}\mathcal{F}\{f_X\}.\mathcal{F}\{f_Y\} \\ &= F^{-1}\{exp[-j\omega\mu_X]exp\left[-\frac{\sigma_X^2\omega^2}{2}\right] exp[-j\omega\mu_Y]exp\left[-\frac{\sigma_Y^2\omega^2}{2}\right]\} \\ &= F^{-1}\{exp[-j\omega(\mu_X+\mu_Y)]exp\left[-\frac{(\sigma_X^2+\sigma_Y^2)\omega^2}{2}\right]\} \\ &= N(z;\mu_X+\mu_Y,\sigma_X^2+\sigma_Y^2) \end{aligned}$$

Consider a random variable Z is Gaussian distributed with parameters μ and σ, (abbreviated as $N(\mu;\sigma^2)$, if it is continuous with p.d.f. (probability density function)

$$\phi(Z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

Let Z_1, Z_2, Z_3, Z_n be i.i.d. standard Gaussians, , then their sum is

$$\begin{aligned} W &= \sum_{i=1}^{n} Z_i \\ &= \sum_{i=1}^{2} Z_i = Z_1 + Z_2 \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{z_1^2}{2}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{z_2^2}{2}} \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{z_1^2+z_2^2}{2}} \end{aligned}$$