Chapter 1

Random-Process[?, ?]

1.1 Introduction

Random Processes:

- Any physical quantity that varies with time is a signal.
- Examples of signals are an electrocardiogram (ECG), and an electroencephalogram (EEG)signals.
- There are two types signals which are
 - 1. Continuous time signals
 - 2. Discrete time signals
- Continuous time signals have the time variable t takes values from $-\infty$ to ∞ or in an interval between t_1 to t_2 .
- Continuous time signals are indicated as x(t), y(t), z(t) and so on.
- A discrete-time signal is a set of measurements taken sequentially in time (e.g., at every millisecond).
- Each measurement point is usually called a sample, and a discrete-time signal is indicated by by x(n), y(n), z(n), where the index n is an integer that points to the order of the measurements in the sequence.
- A random process is a time-varying function that contains the outcome of a random experiment for each time instant, X(t).
- A random process is a time varying function, e.g., a signal.
- A random process consists of infinite number of random variables.
- Random Process are of two types
 - 1. Continuous random process
 - 2. Discrete random process
- Real random process also called stochastic process
- The collection of all possible sample functions of X(t) is called an ensemble of X(t).



Figure 1.1: An Ensemble of Signals

Probability Distribution and Density Functions 1.2

The cdf of a random process is defined as

$$F_X(x,t) = P\{x(t) \le x\}$$

The pdf of a random process is defined as

$$f_X(x,t) = \frac{dF_X(x,t)}{dx}$$

The bivariate cdf of a random process is defined as

$$F_{X(t_1)X(t_2)(x,1x_2)} = P\{X(t_1) \le x_1, X(t_2) \le x_2\}$$

The bivariate pdf of a random process is defined as

$$f_{X(x_1,t_1)(x_2,t_2)}(x,1x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F_{X(t_1)X(t_2)}$$

The bivariate cdf of a nth order multivariate random process is defined as

$$F_{X(t_1)X(t_2)(x,1x_2)\dots(x_n,t_n)} = P\{X(t_1) \le x_1, X(t_2) \le x_2\dots X(t_n) \le x_n\}$$

Similarly nth order multivariate density functions of order n exist

$$f_{X(x_1,t_1)(x_2,t_2)\dots(x_n,t_n)}(x,1x_2 \ x_n) = \frac{\partial^2}{\partial x_1 \partial x_2} F_{X(t_1)X(t_2)}(x_1,1x_2 \ x_n)$$

The mean of a random process is defined as

$$E[X(t)] = \int_{-\infty}^{\infty} x f_X(x,t) dx$$

The mean square μ_X of a random process is defined as

$$\overline{X^2(t)} = E[X^2(t) = \int_{-\infty}^{\infty} x^2 f_X(x,t) dx$$



The variance σ_X^2 of a random process is defined as

$$\begin{aligned} \sigma_X^2 &= E[(X - \mu_X)^2] \\ &= E[X^2 - 2\mu_X X + \mu_X^2] \\ &= E[X^2] - 2\mu_X E[X] + \mu_X^2 \\ &= E[X^2] - 2\mu_X^2 + \mu_X^2 \\ &= E[X^2] - \mu_X^2 \end{aligned}$$

1.3Stationary

A random process $\{X(t)\}$ is stationary if the pdfs and its statistical properties are invariant with changes in time. For example, for a stationary process, X(t) and $X(t+\Delta)$ have the same probability distributions.

$$F_{X(t)}(x) = F_{X(t+\Delta)}(x)$$
 for all $t, t+\Delta$

Stationary of order one: Consider a pdf of random process X(t) is

$$f_{X(t+t_s)}(x) = f_{X(t)}(x) = f_X(x)$$

The pdf is independent of any time shift hence its mean of the random process X(t) is also a constant

$$\mu_{X(t)}(x) = \mu_X$$

The variance of the random process X(t) is also a constant

$$\sigma_{X(t)}^2 = \sigma_X^2$$

Stationary of order two: Consider a pdf of random process X(t) is

$$f_{X(t_1+t_s)X(t_2+t_s)}(x_1, x_2) = f_{X(t_1)X(t_2)}(x_1, x_2)$$

Let

$$t_{1} + t_{s} = t$$

$$t_{s} = t - t_{1}$$

$$t_{2} + t_{s} = t_{2} + t - t_{1}$$

$$t_{2} + t_{s} = t + (t_{2} - t_{1})$$

$$= t + \tau$$

The mean of the random process X(t) is also a constant

$$E[X(t)] = \mu_X$$

$$E[X(t_1)X(t_2)] = E[X(t)X(t+\tau)]$$

The correlation of a wide sense stationer is independent of time, it depends upon τ . When the time difference is 0 i.e., $\tau = t_2 - t_1 = 0 \Rightarrow t_1 = t_2 = t_1$

$$E[X(t)X(t)] = E[X^{2}(t)] = \sigma_{X}^{2} + \mu_{X}^{2}$$

The above relation is a autocorrelation $R_X(\tau)$ of a random process X(t)

$$R_X(\tau) = E[X^2(t)] = \sigma_X^2 + \mu_X^2$$

If a random process X(t) has first and second order stationary, then it called Wide-Sense Stationary Random Processes

Wide-Sense Stationary Random Processes A continuous-time random process X(t) is wide-sense stationary (WSS) if it follows the following properties

1. The mean is independent of time t

$$E[X(t)] = \mu_{X(t)} = \mu_X = constant$$

The variance of the random process X(t) is also a constant

$$\sigma_{X(t)}^2 = \sigma_X^2$$

2. The autocorrelation function only depends on time difference

$$R_X(\tau) = E[X(t)X(t)] = E[X^2(t)] = \sigma_X^2 + \mu_X^2$$

A discrete-time random process $\{X(n), n \in Z\}$ is weak-sense stationary or wide-sense stationary (WSS) if

- 1. $\mu_X(n) = \mu_X$ for all $n \in Z$
- 2. $R_X(n_1, n_2) = R_X(n_1 n_2)$ for all $n_1, n_2 \in Z$



Correlation Functions: 1.4

The correlation functions between two random variables is a measure of the similarity between the variables.

There are two types of correlation functions

- 1. Auto-Correlation Functions
- 2. Cross-Correlation Functions

1.4.1**Auto-Correlation Functions:**

The correlation of a signal X(t) with itself is called as autocorrelation. This is denoted as

$$R_X(\tau) = E[X(t)X(t+\tau)]$$

where t and τ are arbitrary. When $\tau = 0$ the autocorrelation function is the average of the random process squared. It is also called average power.

$$R_X(0) = E[X(t)X(t)] = E[X^2(t)] = \sigma_X^2 + \mu_X^2$$

For $-\tau$

$$R_X(-\tau) = E[X(t)X(t-\tau)]$$

Let $t' = t - \tau$

$$R_X(-\tau) = E[X(t'+\tau)X(t')]$$

= $E[X(t')X(t'+\tau)]$

$$R_X(-\tau) = R_X(\tau)$$

The autocorrelation function is an even function of τ .

$$E[X_i X_j] = \begin{cases} E[X_i^2] = E[X^2] = \mu_X^2 + \sigma_X^2 & j = i \\ E[X_i X_i] = \mu_X^2 & j \neq i \end{cases}$$

The bounds on auto-correlation function is

$$E[\{X(t) \pm X(t+\tau)\}^2] \ge = 0$$

$$E[\{X^2(t) \pm 2X(t)X(t+\tau) + X^2(t+\tau)\}^2] \ge = 0$$

By performing expectations

$$R_X(0) \pm 2R_X(\tau) + R_X(0) \ge 0$$
$$|R_X(\tau)| \le R_X(0)$$

If X(t) is a random process with non zero mean, then it is defined as *auto-covariance function* $C_X(\tau)$

$$C_X(\tau)R_X(0) = E[\{X(t) - \mu_X\}\{X(t+\tau) - \mu_X\}]$$

= $R_X(\tau) - \mu_X^2$

The power spectral density of is defined as

$$\int_{-\infty}^{\infty} R_X(\tau) e^{-j\omega\tau} d\tau \ge 0 \text{ for all } \omega$$



Properties of Auto Correlation Functions

• $R_X(\tau)$ is bounded as

$$|R_X(\tau)| \le R_X(0)$$

• Auto Corrrelation function is even symmetry

$$R_X(-\tau) = R_X(\tau)$$

• The mean value of the random process is obtained using autocorrelation using the following relation

$$E[X^{2}(t)] = \overline{X^{2}(t)} = \sigma_{X}^{2} + \mu_{X}^{2} = R_{X}(0)$$

1.4.2**Cross-Correlation Functions:**

Consider a two random process X(t) and Y(t) are wide sense stationary. When we consider both the random process it is called as jointly wide sense stationary. Then their cross-correlation function is defined as

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)]$$

$$R_{YX}(\tau) = E[Y(t)X(t+\tau)]$$

where t and τ are arbitrary. The order of the subscript is

$$R_{XY}(-\tau) = R_{YX}(\tau)$$

If Y(t) is periodic with period T, then

$$R_{XY}(\tau + T) = R_{XY}(\tau)$$

If X(t) is periodic with T, then

$$R_{YX}(\tau + T) = R_{YX}(\tau)$$

The bounds on cross-correlation function is

$$E[\{X(t) \pm kY(t+\tau)\}^2] \ge = 0$$

$$E[\{X^2(t) \pm 2kX(t)Y(t+\tau) + k^2Y^2(t+\tau)\}^2] \ge = 0$$

By performing expectations

$$R_X(0) \pm 2kR_{XY}(\tau) + k^2R_Y(0) \ge 0$$

If k = 1

$$R_X(0) \pm 2R_{XY}(\tau) + R_Y(0) \geq 0$$

Then it becomes

$$|R_{XY}(\tau)| \leq = \frac{1}{2}[R_X(0) + R_Y(0)]$$

If k is a positive and real constant then

$$k^2 R_Y(0) + 2k R_{XY}(\tau) + R_X(0) \ge 0$$

The quadratic will be never negative if it does not have real roots. If its discriminant is

$$4R_{XY}^2(\tau) - 4R_X(0)R_Y(0) \leq 0$$



$$|R_{XY}(\tau)| \leq \sqrt{R_X(0)R_Y(0)}$$

For example for a quadratic equation

$$ax^2 + bx + c^2 = 0$$

then its discriminant is $b^2 - 4ac$

The geometric mean is

$$\sqrt{R_X(0)R_Y(0)} \leq \frac{1}{2}[R_X(0) + R_Y(0)]$$

A cross correlation function is

$$C_{XY}(\tau) = E[(X(t) - \mu_X)(Y(t + \tau) - \mu_Y)]$$

= $R_{XY}(\tau) - \mu_X \mu_Y$

$$C_{YX}(\tau) = E[(Y(t) - \mu_Y)(X(t + \tau) - \mu_X)]$$

= $R_{YX}(\tau) - \mu_Y \mu_X$

Addition and Subtraction

$$sin(A + B) = sinAcosB - cosAsinB$$

$$sin(A - B) = sinAcosB + cosAsinB$$

$$cos(A + B) = cosAcosB - sinAsinB$$

$$cos(A - B) = cosAcosB + sinAsinB$$

$$tan(A + B) = \frac{tanA + tanB}{1 - tanAtanB}$$

$$tan(A - B) = \frac{tanA - tanB}{1 + tanAtanB}$$

Product Identities

$$sinAcosB = \frac{1}{2}(sin(A+B) + sin(A-B))$$

$$cosAsinB = \frac{1}{2}(sin(A+B) - sin(A-B))$$

$$cosAcosB = \frac{1}{2}(cos(A+B) + cos(A-B))$$

$$sinAsinB = \frac{1}{2}(cos(A-B) - cos(A+B))$$

Example 4.2 The random process described by

$$Y(t) = A\cos(\omega_c t + \Theta)$$

where A and $\omega_c t$ are constants and Θ is a random variable distributed uniformly between $\pm \pi$ Find the pdf of random variable Θ , mean and autocorrelation function of Y

Solution:



The pdf of random variable Θ

$$f_{\Theta}(\theta) = \frac{1}{b-a} = \frac{1}{\pi - (-\pi)}$$
$$= \frac{1}{2\pi}$$
$$f_{\Theta}(\theta) = \begin{cases} \frac{1}{2\pi} & -\pi < \theta < \pi\\ 0 & otherwise \end{cases}$$

The mean of random variable Y

$$\mu_Y = \int_{-\pi}^{\pi} A\cos(\omega_c t + \theta) \frac{1}{2\pi} d\theta$$

= $\frac{A}{2\pi} \int_{-\pi}^{\pi} \cos\omega_c t \cos\theta - \sin\omega_c t \sin\theta d\theta$
= $\frac{A}{2\pi} [\cos\omega_c t \sin\theta + \sin\omega_c t \cos\theta]_{-\pi}^{\pi}$
= $\frac{A}{2\pi} [0 + \sin\omega_c t (-1 - (-1))]$
= 0

Autocorrelation function $R_Y(\tau)$

$$\begin{split} E[Y(t)Y(t+\tau)] &= \int_{-\pi}^{\pi} \left(\frac{1}{2\pi}A\cos(\omega_{c}t+\theta)\right) \left(\frac{1}{2\pi}A\cos(\omega_{c}(t+\tau)+\theta)\right) d\theta \\ &= \frac{A^{2}}{2\pi} \int_{-\pi}^{\pi}\cos(\omega_{c}t+\theta)\cos(\omega_{c}(t+\tau)+\theta) d\theta \\ &= \frac{A^{2}}{2\pi} \int_{-\pi}^{\pi} [\cos(2\omega_{c}t+\omega_{c}\tau+2\theta)+\cos\omega_{c}\tau] d\theta \\ &= \frac{A^{2}}{2\pi} \int_{-\pi}^{\pi} [\cos(2\omega_{c}t+\omega_{c}\tau+2\theta)+\cos\omega_{c}\tau] d\theta \\ &\qquad x = 2\omega_{c}t+\omega_{c}\tau+2\theta \\ &\qquad \frac{dx}{d\theta} = 0+0+2 \\ &\qquad d\theta = \frac{dx}{2} \end{split}$$

Limits, when

 $x = 2\omega_c t + \omega_c \tau + 2\pi$

when $\theta = -\pi$

$$x = 2\omega_c t + \omega_c \tau - 2\pi$$

$$\int_{-\pi}^{\pi} [\cos(2\omega_{c}t + \omega_{c}\tau + 2\theta)] d\theta = \int_{2\omega_{c}t + \omega_{c}\tau - 2\pi}^{2\omega_{c}t + \omega_{c}\tau - 2\pi} (\cos x) \frac{dx}{2}$$

$$= \frac{1}{2} [\sin x]_{2\omega_{c}t + \omega_{c}\tau - 2\pi}^{2\omega_{c}t + \omega_{c}\tau - 2\pi}$$

$$= \frac{1}{2} [\sin(2\omega_{c}t + \omega_{c}\tau + 2\pi) - \sin(2\omega_{c}t + \omega_{c}\tau - 2\pi)]$$

$$= \frac{1}{2} [\sin(2\omega_{c}t + \omega_{c}\tau + 2\pi) - \sin(2\omega_{c}t + \omega_{c}\tau - 2\pi)]$$

$$= \frac{1}{2} [\sin(2\omega_{c}t + \omega_{c}\tau + 2\pi) - \sin(2\omega_{c}t + \omega_{c}\tau - 2\pi)]$$

$$= \frac{1}{2} [\sin(2\omega_{c}t + \omega_{c}\tau) - \sin(2\omega_{c}t + \omega_{c}\tau)]$$

$$= 0$$

$$\int_{-\pi}^{\pi} [\cos\omega_c \tau] d\theta = \cos\omega_c \tau [\theta]_{-\pi}^{\pi}$$
$$= 2\pi \cos\omega_c \tau$$
$$E[Y(t)Y(t+\tau)] = \frac{A^2}{2\pi} 2\pi \cos\omega_c \tau$$

$$= \frac{A^2}{2} \cos \omega_c \tau$$

The ensemble variance is constant

$$\sigma_Y^2 = R_Y(0) = \frac{A^2}{2}$$

From the above discussions it is observed that

1. E[Y(t)] = 0=Constant and

2. $E[Y(t)Y(t+\tau)] = R_Y(\tau)$, it is independent of absolute time hence

Y(t) is a wide sense stationary.

Example 4.6 Consider two random process X(t) and Y(t) which are independent, jointly wide sense stationary random process described by

$$X(t) = A\cos(\omega_1 t + \Theta_1)$$

$$Y(t) = B\cos(\omega_2 t + \Theta_2)$$

where A, B and $\omega_1 t \omega_2 t$ are constants and Θ_1 and Θ_2 are random variable distributed uniformly between $\pm \pi$ Let X(t) and Y(t) are related by

$$W(t)X(t) = X(t)Y(t)$$

Find the autocorrelation function of W

Solution:

The autocorrelation function of X(t) and Y(t) are

$$R_X(\tau) = \frac{A^2}{2} cos\omega_1 \tau$$
$$R_Y(\tau) = \frac{B^2}{2} cos\omega_2 \tau$$

$$R_W(\tau) = \frac{A^2}{2} \cos\omega_1 \tau \frac{B^2}{2} \cos\omega_2 \tau$$

= $\frac{A^2 B^2}{4} \cos\omega_1 \tau \cos\omega_2 \tau$
= $\frac{A^2 B^2}{8} [\cos(\omega_1 + \omega_2)\tau + (\omega_1 - \omega_2)\tau]$



6. The random process described by

$$X(t) = Asin(\omega_c t + \Theta)$$

where A and ω_c are constants and Θ is a random variable uniformly distributed between $\pm \pi$. Is X(t) wide-sense stationary? If not, then why not? If so, then what are the mean and autocorrelation function for the random process? [?]

Solution:

The pdf of random variable Θ

$$f_{\Theta}(\theta) = \frac{1}{b-a} = \frac{1}{\pi - (-\pi)}$$
$$= \frac{1}{2\pi}$$

$$f_{\Theta}(\theta) = \begin{cases} \frac{1}{2\pi} & -\pi < \theta < \pi \\ 0 & otherwise \end{cases}$$

The mean of random variable X

$$\mu_X = \int_{-\pi}^{\pi} Asin(\omega_c t + \theta) \frac{1}{2\pi} d\theta$$

= $\frac{A}{2\pi} \int_{-\pi}^{\pi} sin\omega_c t \cos\theta - \cos\omega_c t \sin\theta d\theta$
= $\frac{A}{2\pi} [sin\omega_c tsin\theta + \cos\omega_c tcos\theta]_{-\pi}^{\pi}$
= $\frac{A}{2\pi} [0 + sin\omega_c t(-1 - (-1))]$
= 0

Autocorrelation function $R_X(\tau)$

$$\begin{split} E[X(t)X(t+\tau)] &= \int_{-\pi}^{\pi} \frac{1}{2\pi} \left(Asin(\omega_c t+\theta)) \left(Asin(\omega_c (t+\tau)+\theta) \right) d\theta \\ &= \frac{A^2}{2\pi} \int_{-\pi}^{\pi} sin(\omega_c t+\theta) sin(\omega_c (t+\tau)+\theta) d\theta \\ &= \frac{A^2}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} [cos\omega_c \tau - cos(2\omega_c t+\omega_c \tau+2\theta)] d\theta \\ &= \frac{A^2}{4\pi} \int_{-\pi}^{\pi} [cos\omega_c \tau - cos(2\omega_c t+\omega_c \tau+2\theta)] d\theta \end{split}$$

 $x = 2\omega_c t + \omega_c \tau + 2\theta$ $\frac{dx}{d\theta} = 0 + 0 + 2$ $\frac{d\theta}{d\theta} = \frac{dx}{2}$ Limits, when $\theta = \pi$ $x = 2\omega_c t + \omega_c \tau + 2\pi$ when $\theta = -\pi$ $x = 2\omega_c t + \omega_c \tau - 2\pi$



$$\int_{-\pi}^{\pi} [\cos(2\omega_c t + \omega_c \tau + 2\theta)] d\theta = \int_{2\omega_c t + \omega_c \tau + 2\pi}^{2\omega_c t + \omega_c \tau + 2\pi} (\cos x) \frac{dx}{2}$$

$$= \frac{1}{2} [\sin x]_{2\omega_c t + \omega_c \tau + 2\pi}^{2\omega_c t + \omega_c \tau + 2\pi}$$

$$= \frac{1}{2} [\sin(2\omega_c t + \omega_c \tau + 2\pi) - \sin(2\omega_c t + \omega_c \tau - 2\pi)]$$

$$= \frac{1}{2} [\sin(2\omega_c t + \omega_c \tau + 2\pi) - \sin(2\omega_c t + \omega_c \tau - 2\pi)]$$

$$= \frac{1}{2} [\sin(2\omega_c t + \omega_c \tau) - \sin(2\omega_c t + \omega_c \tau - 2\pi)]$$

$$= 0$$

$$\int_{-\pi}^{\pi} [\cos\omega_c \tau] d\theta = \cos\omega_c \tau [\theta]_{-\pi}^{\pi}$$
$$= 2\pi \cos\omega_c \tau$$

$$E[X(t)X(t+\tau)] = \frac{A^2}{4\pi} 2\pi \cos\omega_c \tau$$
$$R_X(\tau) = \frac{A^2}{2} \cos\omega_c \tau$$

From the above discussions it is observed that

- 1. E[X(t)] = 0=Constant and
- 2. $E[X(t)X(t+\tau)] = R_X(\tau)$, it is independent of absolute time hence

X(t) is a wide sense stationary.

7. The random process described by

$$X(t) = A\cos(\omega_c t + \phi + \Theta)$$

where A ω_c and ϕ are constants and Θ is a random variable uniformly distributed between $\pm \pi$. Is X(t) wide-sense stationary? If not, then why not? If so, then what are the mean and autocorrelation function for the random process? [?]

Solution:

The pdf of random variable Θ

$$f_{\Theta}(\theta) = \frac{1}{b-a} = \frac{1}{\pi - (-\pi)}$$
$$= \frac{1}{2\pi}$$

$$f_{\Theta}(\theta) = \begin{cases} \frac{1}{2\pi} & -\pi < \theta < \pi \\ 0 & otherwise \end{cases}$$

The mean of random variable X



$$\mu_X = \int_{-\pi}^{\pi} \cos(\omega_c t + \phi + \Theta) \frac{1}{2\pi} d\theta$$

$$\begin{aligned} x &= \omega_c t + \phi + \theta \\ \frac{dx}{d\theta} &= 0 + 0 + d\theta \\ d\theta &= dx \end{aligned}$$

Limits, when $\theta = \pi$

$$x = \omega_c t + \phi + \pi$$

when $\theta = -\pi$

$$x = \omega_c t + \phi - \pi$$

$$\mu_X = \int_{-\pi}^{\pi} \cos(\omega_c t + \phi + \Theta) \frac{1}{2\pi} d\theta$$

$$= \frac{A}{2\pi} \int_{\omega_c t + \phi + \pi}^{\omega_c t + \phi + \pi} \cos x dx$$

$$= \frac{A}{2\pi} [\sin x]_{\omega_c t + \phi - \pi}^{\omega_c t + \phi + \pi}$$

$$= \frac{A}{2\pi} [\sin(\omega_c t + \phi + \pi) - \sin(\omega_c t + \phi - \pi)]$$

$$= \frac{A}{2\pi} [-\sin(\omega_c t + \phi) + \sin(\omega_c t + \phi)]$$

$$= 0$$

Autocorrelation function $R_X(\tau)$

Limits, when

 $x = 2\omega_c t + \omega_c \tau + 2\pi$

when $\theta = -\pi$

 $x = 2\omega_c t + \omega_c \tau - 2\pi$



$$\int_{-\pi}^{\pi} [\cos(2\omega_c t + \omega_c \tau + 2\theta)] d\theta = \int_{2\omega_c t + \omega_c \tau - 2\pi}^{2\omega_c t + \omega_c \tau + 2\pi} (\cos x) \frac{dx}{2}$$

$$= \frac{1}{2} [\sin x]_{2\omega_c t + \omega_c \tau - 2\pi}^{2\omega_c t + \omega_c \tau + 2\pi}$$

$$= \frac{1}{2} [\sin(2\omega_c t + \omega_c \tau + 2\pi) - \sin(2\omega_c t + \omega_c \tau - 2\pi)]$$

$$= \frac{1}{2} [\sin(2\omega_c t + \omega_c \tau + 2\pi) - \sin(2\omega_c t + \omega_c \tau - 2\pi)]$$

$$= \frac{1}{2} [\sin(2\omega_c t + \omega_c \tau) - \sin(2\omega_c t + \omega_c \tau - 2\pi)]$$

$$= 0$$

$$\int_{-\pi}^{\pi} [\cos\omega_c \tau] d\theta = \cos\omega_c \tau [\theta]_{-\pi}^{\pi}$$
$$= 2\pi \cos\omega_c \tau$$

$$E[X(t)X(t+\tau)] = R_X(\tau) = \frac{A^2}{4\pi} 2\pi \cos\omega_c \tau = \frac{A^2}{2} \cos\omega_c \tau$$

From the above discussions it is observed that

- 1. E[X(t)] = 0=Constant and
- 2. $E[X(t)X(t+\tau)] = R_X(\tau)$, it is independent of absolute time hence

X(t) is a wide sense stationary.

8. The random process described by

$$X(t) = A\cos(\omega_c t + \Theta) + B$$

where A, B and ω_c are constants and Θ is a random variable uniformly distributed between $\pm \pi$. Is X(t) wide-sense stationary? If not, then why not? If so, then what are the mean and autocorrelation function for the random process? [?]

Solution:

The pdf of random variable Θ

$$f_{\Theta}(\theta) = \frac{1}{b-a} = \frac{1}{\pi - (-\pi)}$$
$$= \frac{1}{2\pi}$$

$$f_{\Theta}(\theta) = \begin{cases} \frac{1}{2\pi} & -\pi < \theta < \pi \\ 0 & otherwise \end{cases}$$

The mean of random variable X

$$\mu_X = E[X(t)] = E[A\cos(\omega_c t + \Theta) + B]$$

=
$$\int_{-\pi}^{\pi} A\cos(\omega_c t + \theta) \frac{1}{2\pi} d\theta + B \int_{-\pi}^{\pi} \frac{1}{2\pi} d\theta$$

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$$x = \omega_c t + \theta$$
$$dx = 0 + d\theta$$
$$d\theta = dx$$

Limits, when $\theta = \pi$

 $x = \omega_c t + \pi$

when $\theta = -\pi$

 $x = \omega_c t - \pi$

$$\mu_X = \int_{-\pi}^{\pi} A\cos(\omega_c t + \theta) \frac{1}{2\pi} d\theta$$

= $\frac{A}{2\pi} \int_{\omega_c t - \pi}^{\omega_c t + \pi} \cos x dx$
= $\frac{A}{2\pi} [\sin x]_{\omega_c t - \pi}^{\omega_c t + \pi}$
= $\frac{A}{2\pi} [\sin(\omega_c t + \pi) - \sin(\omega_c t - \pi)]$
= $\frac{A}{2\pi} [-\sin(\omega_c t) + \sin(\omega_c t)]$
= 0

$$B \int_{-\pi}^{\pi} \frac{1}{2\pi} d\theta = \frac{B}{2\pi} [\theta]_{-\pi}^{\pi}$$
$$= \frac{B}{2\pi} [2\pi]$$
$$= B$$

Autocorrelation function $R_X(\tau)$

$$cosAcosB = \frac{1}{2}(cos(A+B)+cos(A-B))$$

$$\begin{split} E[X(t)X(t+\tau)] &= E[(A\cos(\omega_c t + \phi + B)) \left(A\cos(\omega_c (t+\tau) + B))\right] \\ &= E[\left(A^2\cos(\omega_c t + \phi)\right) \left(\cos(\omega_c (t+\tau) + \phi)\right) + AB \left(\cos(\omega_c (t+\tau) + \phi)\right) + AB \left(\cos(\omega_c t + \phi)\right) + B \right) \\ &= \frac{A^2}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} \left[\cos(2\omega_c t + \omega_c \tau + 2\theta) + \cos(\omega_c \tau)\right] d\theta + 0 + 0 + B^2 \\ &= \frac{A^2}{2} \cos(\omega_c \tau) + B^2 \end{split}$$

$$E[X(t)X(t+\tau)] = \frac{A^2}{2}cos(\omega_c\tau) + B^2$$
$$R_X(\tau) = \frac{A^2}{2}cos(\omega_c\tau) + B^2$$

From the above discussions it is observed that

- 1. E[X(t)] = B=Constant and
- 2. $E[X(t)X(t+\tau)] = R_X(\tau)$, it is independent of absolute time hence



X(t) is a wide sense stationary.

10. The random process described by

$$X(t) = A^2 \cos^2(\omega_c t + \Theta)$$

where A and ω_c are constants and Θ is a random variable uniformly distributed between $\pm \pi$. Is X(t) wide-sense stationary? If not, then why not? If so, then what are the mean and autocorrelation function for the random process? [?]

Solution:

The pdf of random variable Θ

$$f_{\Theta}(\theta) = \frac{1}{b-a} = \frac{1}{\pi - (-\pi)}$$
$$= \frac{1}{2\pi}$$

$$f_{\Theta}(\theta) = \begin{cases} \frac{1}{2\pi} & -\pi < \theta < \pi \\ 0 & otherwise \end{cases}$$

The mean of random variable X

$$\mu_X = E[X(t)] = E[A^2 cos^2(\omega_c t + \Theta)]$$

= $\frac{A^2}{2}E[1 + cos(2\omega_c t + 2\Theta)]$
= $\frac{A^2}{2} + \frac{A^2}{2} \int_{-\pi}^{\pi} cos(2\omega_c t + 2\theta)d\theta$

$$x = 2\omega_c t + 2\theta$$
$$dx = 0 + 2d\theta$$
$$d\theta = \frac{dx}{2}$$

Limits, when $\theta = \pi$

$$x = 2\omega_c t + 2\pi$$

when $\theta = -\pi$

$$x = 2\omega_c t - 2\pi$$

$$= \int_{-\pi}^{\pi} \cos(2\omega_c t + 2\theta) \frac{1}{2\pi} d\theta$$

$$= \frac{A^2}{4\pi} \int_{2\omega_c t - 2\pi}^{2\omega_c t + 2\pi} \cos x dx$$

$$= \frac{A^2}{4\pi} [\sin x]_{2\omega_c t - 2\pi}^{2\omega_c t + 2\pi}$$

$$= \frac{A^2}{4\pi} [\sin(2\omega_c t + 2\pi) - \sin(2\omega_c t - 2\pi)]$$

$$= \frac{A^2}{4\pi} [\sin(2\omega_c t) - \sin(2\omega_c t)] = 0$$

$$= \frac{A^2}{2}$$

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 μ_X



Autocorrelation function $R_X(\tau)$

$$\begin{split} E[X(t)X(t+\tau)] &= E[\left(A^2 cos^2(\omega_c t+\Theta)\right) \left(A cos(\omega_c(t+\tau)+B)A^2 cos^2(\omega_c t+\Theta)\right)] \\ &= \frac{A^2}{4} E[\{1+cos(2\omega_c t+2\Theta)\}\{1+cos(2\omega_c(t+\tau)+2\Theta)\}] \\ &= \frac{A^2}{4}\{1+\int_{-\pi}^{\pi} cos(2\omega_c t+2\theta)\frac{1}{2\pi}d\theta + \int_{-\pi}^{\pi} cos(2\omega_c(t+\tau)+2\theta)\frac{1}{2\pi}d\theta\} \\ &+ \int_{-\pi}^{\pi} cos(2\omega_c t+2\theta)cos(2\omega_c(t+\tau)+2\theta)\frac{1}{2\pi}d\theta \\ &= \frac{A^2}{4}\{1+0+0+\frac{1}{4\pi}\int_{-\pi}^{\pi} [cos(4\omega_c t+2\omega_c \tau+4\theta)+cos(2\omega_c \tau)]d\theta\} \\ &= \frac{A^2}{4}\{1+\frac{1}{2}cos(2\omega_c \tau)\} \end{split}$$

$$E[X(t)X(t+\tau)] = \frac{A^2}{4} \{1 + \frac{1}{2}cos(2\omega_c\tau)\}$$
$$R_X(\tau) = \frac{A^2}{4} \{1 + \frac{1}{2}cos(2\omega_c\tau)\}$$

From the above discussions it is observed that

- 1. $E[X(t)] = \frac{A^2}{2}$ = Constant and
- 2. $E[X(t)X(t+\tau)] = R_X(\tau)$, it is independent of absolute time hence

X(t) is a wide sense stationary.

12. The random process described by

$$X(t) = A\cos(\omega_c t + \Theta) + B\cos(\omega_s t)$$

where A, B ω_c , and $\omega_s \omega_c \neq \omega_s$ are constants and Θ is a random variable uniformly distributed between $\pm \pi$. Is X(t) wide-sense stationary? If not, then why not? If so, then what are the mean and autocorrelation function for the random process? [?]

Solution:

The pdf of random variable Θ

$$f_{\Theta}(\theta) = \frac{1}{b-a} = \frac{1}{\pi - (-\pi)}$$
$$= \frac{1}{2\pi}$$

$$f_{\Theta}(\theta) = \begin{cases} rac{1}{2\pi} & -\pi < heta < \pi \\ 0 & otherwise \end{cases}$$

The mean of random variable X

$$\mu_X = E[X(t)] = E[A\cos(\omega_c t + \Theta) + B\cos(\omega_s t)]$$

= $A \int_{-\pi}^{\pi} \cos(\omega_c t + 2\theta) \frac{1}{2\pi} d\theta + B\cos(\omega_s t)$
= $B\cos(\omega_s t)$



From the above discussions it is observed that X(t) is not a wide sense stationary, because its mean is not a constant.

13. The random process described by

 $X(t) = A\cos(\omega_c t + \Theta)$

where A and ω_c , are constants and Θ is a random variable uniformly distributed between $\pm \frac{\pi}{2}$. Is X(t) wide-sense stationary? If not, then why not? If so, then what are the mean and autocorrelation function for the random process? [?]

Solution:

The pdf of random variable Θ

$$f_{\Theta}(\theta) = \frac{1}{b-a} = \frac{1}{\pi/2 - (-\pi/2)}$$

= $\frac{1}{\pi}$

$$f_{\Theta}(\theta) = \begin{cases} \frac{1}{\pi} & -\pi/2 < \theta < \pi/2 \\ 0 & otherwise \end{cases}$$

The mean of random variable X

$$\mu_X = E[X(t)] = E[A\cos(\omega_c t + \Theta)\frac{1}{\pi}]$$
$$= A \int_{-\pi/2}^{\pi/2} \cos(\omega_c t + \theta)\frac{1}{\pi}d\theta$$

$$\begin{array}{rcl} x & = & \omega_c t + \theta \\ d\theta & = & dx \end{array}$$

Limits, when $\theta = \pi/2$

$$x = \omega_c t + \pi/2$$

when $\theta = -\pi/2$

$$x = \omega_c t - \pi/2$$

$$\mu_X = A \int_{-\pi/2}^{\pi/2} \cos(\omega_c t + \theta) \frac{1}{\pi} d\theta$$

= $\frac{A}{\pi} \int_{\omega_c t - \pi/2}^{\omega_c t + \pi/2} \cos(x) dx$
= $\frac{A}{\pi} \cos(\omega_c t + \pi/2) - \cos(\omega_c t - \pi/2)$
= $\frac{2A}{\pi} \cos(\omega_c t)$

From the above discussions it is observed that X(t) is not a wide sense stationary, because its mean is not a constant.

14. The random process described by

$$X(t) = A\cos(\omega_c t + \Theta)$$

where A and ω_c , are constants and Θ is a random variable uniformly distributed between 0 and π . Is X(t) wide-sense stationary? If not, then why not? If so, then what are the mean and autocorrelation function for the random process? [?]

Solution:

The pdf of random variable Θ

$$f_{\Theta}(\theta) = \frac{1}{b-a} = \frac{1}{\pi}$$
$$= \frac{1}{\pi}$$
$$f_{\Theta}(\theta) = \begin{cases} \frac{1}{\pi} & 0 < \theta < \pi\\ 0 & otherwise \end{cases}$$

The mean of random variable X

$$\mu_X = E[X(t)] = E[A\cos(\omega_c t + \Theta)\frac{1}{\pi}]$$
$$= A \int_0^{\pi} \cos(\omega_c t + \theta)\frac{1}{\pi}d\theta$$
$$x = \omega_c t + \theta$$

 $d\theta = dx$

Limits, when $\theta = 0$

x $= \omega_c t$

when $\theta = \pi$

$$x = \omega_c t + \pi$$

$$\mu_X = A \int_0^{\pi} \cos(\omega_c t + \theta) \frac{1}{\pi} d\theta$$
$$= \frac{A}{\pi} \int_{\omega_c t}^{\omega_c t + \pi} \sin(x) dx$$
$$= \frac{A}{\pi} \sin(\omega_c t + \pi) - \sin(\omega_c t)$$
$$= \frac{-2A}{\pi} \sin(\omega_c t)$$

From the above discussions it is observed that X(t) is not a wide sense stationary, because its mean is not a constant.

15. A random process described by

X(t) = V



where V is a random variable uniformly distributed between 0 and 4. That means that each realization of X(t) = V is constant v, that the constant varies from one realization to the next, and that the variation is described as uniformly distributed between 0 and 4. Is X(t) widesense stationary? If not, then why not? If so, then what are the mean and autocorrelation function for the random process? [?]

Solution:

The pdf of random variable X

$$f_V(v) = \frac{1}{b-a} = \frac{1}{4-(0)}$$
$$= \frac{1}{4}$$
$$f_X(x) = \begin{cases} \frac{1}{4} & 0 < \theta < 4\\ 0 & otherwise \end{cases}$$

The mean of random variable X

$$\mu_X = E[X(t)] = E[V] = \frac{1}{4} \int_0^4 v \, dv = \frac{1}{4} \left[\frac{v^2}{2} \right]_0^4 = \frac{1}{8} \left[4^2 - 0 \right]$$
$$= 2$$

$$E[X(t)X(t+\tau)] = E[V^2] = \frac{1}{4} \int_0^4 v^2 dv = \frac{1}{4} \left[\frac{v^3}{3}\right]_0^4$$
$$= \frac{1}{12} \left[4^3 - 0\right]$$
$$= \frac{16}{3}$$

From the above discussions it is observed that

- 1. E[X(t)] = 2=Constant and
- 2. $E[X(t)X(t+\tau)] = R_X(\tau) = \frac{16}{3}$, it is independent of absolute time hence

X(t) is a wide-sense stationary.

16. A random process described by

$$X(t) = V$$

where V is a random variable uniformly distributed between 0 and 2. That means that each realization of X(t) = V is constant v, that the constant varies from one realization to the next, and that the variation is described as uniformly distributed between 0 and 2. Is X(t) widesense stationary? If not, then why not? If so, then what are the mean and autocorrelation function for the random process? [?]

Solution:

The pdf of random variable X

$$f_V(v) = \frac{1}{b-a} = \frac{1}{2-(0)} = \frac{1}{2}$$



$$f_X(x) = \begin{cases} \frac{1}{2} & 0 < \theta < 2\\ 0 & otherwise \end{cases}$$

The mean of random variable X

$$\mu_X = E[X(t)] = E[V] = \frac{1}{2} \int_0^2 v dv = \frac{1}{2} \left[\frac{v^2}{2} \right]_0^2 = \frac{1}{4} \left[2^2 - 0 \right]$$
$$= 1$$

$$E[X(t)X(t+\tau)] = E[V^2] = \frac{1}{2} \int_0^4 v^2 dv = \frac{1}{2} \left[\frac{v^3}{3}\right]_0^4$$
$$= \frac{1}{6} \left[2^3 - 0\right]$$
$$= \frac{4}{3}$$

From the above discussions it is observed that

- 1. E[X(t)] = 1 = Constant and
- 2. $E[X(t)X(t+\tau)] = R_X(\tau) = \frac{4}{3}$, it is independent of absolute time hence
- X(t) is a wide-sense stationary.

17. A random process described by

X(t) = V

where V is a random variable uniformly distributed between 0 and 3. That means that each realization of X(t) = V is constant v, that the constant varies from one realization to the next, and that the variation is described as uniformly distributed between 0 and 3. Is X(t) widesense stationary? If not, then why not? If so, then what are the mean and autocorrelation function for the random process? [?]

Solution:

The pdf of random variable X

$$f_V(v) = \frac{1}{b-a} = \frac{1}{3-(0)} = \frac{1}{2}$$

$$f_X(x) = \begin{cases} \frac{1}{3} & 0 < \theta < 3\\ 0 & otherwise \end{cases}$$

The mean of random variable X

$$\mu_X = E[X(t)] = E[V] = \frac{1}{3} \int_0^3 v \, dv = \frac{1}{3} \left[\frac{v^2}{2} \right]_0^3 = \frac{1}{6} \left[3^2 - 0 \right]$$
$$= \frac{3}{2}$$



$$E[X(t)X(t+\tau)] = E[V^2] = \frac{1}{3} \int_0^3 v^2 dv = \frac{1}{3} \left[\frac{v^3}{3}\right]_0^3$$
$$= \frac{1}{9} \left[3^3 - 0\right]$$
$$= 3$$

From the above discussions it is observed that

1. $E[X(t)] = \frac{3}{2}$ =Constant and

2.
$$E[X(t)X(t+\tau)] = R_X(\tau) = 3$$
, it is independent of absolute time hence

X(t) is a wide-sense stationary.



Addition of Random Processes: 1.4.3

Consider the addition of two random independent, jointly wide-sense stationary random precess X(t) and Y(t) is

$$W(t) = X(t) + Y(t)$$

The autocorrelation function of the sum W(t) is

$$R_W(\tau) = E[\{X(t) + Y(t)\}\{X(t+\tau) + Y(t+\tau)\}]$$

= $E[X(t)X(t+\tau) + X(t)Y(t+\tau) + Y(t)X(t+\tau) + Y(t)Y(t+\tau)\}]$
= $R_X(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_Y(\tau)$

If X(t) and Y(t) are independent, then the cross-correlation function is

$$R_{XY}(\tau) = R_{YX}(\tau) = \mu_X \mu_Y$$

If X(t) or Y(t) has a zero mean then

$$R_W(\tau) = R_X(\tau) + R_Y(\tau)$$

18. The autocorrelation function for random process Z(t) is

$$R_Z(\tau) = \begin{cases} 50(1 - |\tau|/T) & -T \le t \le T\\ 0 & otherwise \end{cases}$$

A random process X(t) is the sum

$$X(t) = Z(t) + U$$

where U is a random variable with a mean $\mu_U = 4$, a variance $\sigma_U^2 = 25$, and is independent of Z(t). Find the autocorrelation function of X(t)[?]

Solution:

$$E[Z(t)] = 0$$

Autocorrelation function $R_X(\tau)$

$$R_X(\tau) = E[\{Z(t) + U\}\{Z(t + \tau) + U\}]$$

= $E[Z(t)Z(t + \tau) + UZ(t + \tau) + UZ(t) + U^2]$
= $R_Z(\tau) + 0 + 0 + E[U^2]$
= $R_Z(\tau) + \sigma_U^2 + \mu_U^2$
= $R_Z(\tau) + \sigma_U^2 + \mu_U^2$
= $R_Z(\tau) + 25 + 4^2$
= $R_Z(\tau) + 41$

19. The autocorrelation function for random process Z(t) is

$$R_Z(\tau) = \begin{cases} 40(1 - |\tau|/T) & -T \le t \le T\\ 0 & otherwise \end{cases}$$

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A random process X(t) is the sum

$$X(t) = Z(t) + U$$

where U is a random variable with a mean $\mu_U = 5$, a variance $\sigma_U^2 = 15$, and is independent of Z(t). Find the autocorrelation function of X(t) [?]

Solution:

$$E[Z(t)] = 0$$

Autocorrelation function $R_X(\tau)$

$$R_X(\tau) = E[\{Z(t) + U\}\{Z(t + \tau) + U\}]$$

= $E[Z(t)Z(t + \tau) + UZ(t + \tau) + UZ(t) + U^2]$
= $R_Z(\tau) + 0 + 0 + E[U^2]$
= $R_Z(\tau) + \sigma_U^2 + \mu_U^2$
= $R_Z(\tau) + \sigma_U^2 + \mu_U^2$
= $R_Z(\tau) + 15 + 5^2$
= $R_Z(\tau) + 40$

20. The autocorrelation function for random process Z(t) is

$$R_Z(\tau) = \begin{cases} 60(1 - |\tau|/T) & -T \le t \le T\\ 0 & otherwise \end{cases}$$

A random process X(t) is the sum

$$X(t) = Z(t) + U$$

where U is a random variable with a mean $\mu_U = 4$, a variance $\sigma_U^2 = 20$, and is independent of Z(t). Find the autocorrelation function of X(t) [?]

Solution:

$$E[Z(t)] = 0$$

Autocorrelation function $R_X(\tau)$

$$\begin{aligned} R_X(\tau) &= E[\{Z(t) + U\}\{Z(t + \tau) + U\}] \\ &= E[Z(t)Z(t + \tau) + UZ(t + \tau) + UZ(t) + U^2] \\ &= R_Z(\tau) + 0 + 0 + E[U^2] \\ &= R_Z(\tau) + \sigma_U^2 + \mu_U^2 \\ &= R_Z(\tau) + \sigma_U^2 + \mu_U^2 \\ &= R_Z(\tau) + 20 + 4^2 \\ &= R_Z(\tau) + 36 \end{aligned}$$



21. The random process X(t) has the autocorrelation function

$$R_X(\tau) = \begin{cases} 10(1 - |\tau|/\tau_N) & -\tau_N \le \tau \le \tau_N \\ 0 & otherwise \end{cases}$$

The random process Y(t) is independent of X(t) and has the autocorrelation function

$$R_Y(\tau) = \begin{cases} 15(1 - |\tau|/T) & -T \le t \le T\\ 0 & otherwise \end{cases}$$

$$X(t) = Z(t) + U$$

where $T \gg \tau_N$. The random process Z(t) = X(t) + Y(t). For Z(t), find the autocorrelation function, its total power, its dc power, and its ac power. Is Z(t) wide-sense stationary? [?]

Solution:

 $\sigma_X^2 = 10, \mu_X = 0, \sigma_Y^2 = 15, \mu_Y = 0$, Autocorrelation function $R_Z(\tau)$

$$R_Z(\tau) = R_X(\tau) + R_Y(\tau)$$

It is mentioned that $T \gg \tau_N$, $R_Z(\tau)$ is summation of the autocorrelation function of $R_X(\tau)$ and $R_Y(\tau)$, $R_X(\tau)$ is within the limits of $R_Y(\tau)$, hence we have to consider the limits for $R_Z(\tau)$ from $0 \le |\tau| \le \tau_N$ and $\tau_N \leq |\tau| \leq T$ From $0 \leq |\tau| \leq \tau_N$ the $R_Z(\tau)$ is

$$R_Z(\tau) = 15(1 - |\tau|/T) + 10(1 - |\tau|/\tau_N)$$

From $\tau_N \leq |\tau| \leq T$ the $R_Z(\tau)$ is

$$R_Z(\tau) = 15(1 - |\tau|/T)$$

$$R_{Z}(\tau) = \begin{cases} 15(1 - |\tau|/T) + 10(1 - |\tau|/\tau_{N}) & 0 \le |\tau| \le \tau_{N} \\ 15(1 - |\tau|/T) & \tau_{N} \le |\tau| \le T \\ 0 & otherwise \end{cases}$$

From the above discussions it is observed that

1. The mean of $R_Z(\tau) = 0$

2. $R_Z(\tau)$ is independent of absolute time, hence it is wide-sense stationary.

Total power or AC power of $R_Z(\tau)$ is

$$= \sigma_X^2 + \sigma_Y^2 = 10 + 15$$
$$= 25$$



The random process X(t) and Y(t) are jointly wide-sense stationary and they are 22.independent. Given that W(t) = X(t) + Y(t) and

$$R_X(\tau) = 10 \ exp\left(-\frac{|\tau|}{3}\right)$$
$$R_Y(\tau) = \begin{cases} 10\left(\frac{3-|\tau|}{3}\right) & -3 \le \tau \le 0\\ 0 & otherwise \end{cases}$$

For W(t), find its autocorrelation function, its total power, its dc power, and its ac power. Is W(t) wide-sense stationary? |?|

3

Solution:

It is mentioned that X(t) and Y(t) are jointly wide-sense stationary and they are independent. $\sigma_X^2 =$ $10, \mu_X = 0, \sigma_Y^2 = 10, \mu_Y = 0$, Autocorrelation function $R_Z(\tau)$

$$R_W(\tau) = R_X(\tau) + R_Y(\tau)$$

 $R_Z(\tau)$ is summation of the autocorrelation function of $R_X(\tau)$ and $R_Y(\tau)$, $R_X(\tau)$ is within the limits of $R_Y(\tau)$, hence we to consider the limits for $R_Z(\tau)$ from $0 \le |\tau| \le 3$ and other than this limits From $0 \leq |\tau| \leq 3$ the $R_Z(\tau)$ is

$$R_Z(\tau) = 10 \exp\left(-\frac{|\tau|}{3}\right) + 10\left(\frac{3-|\tau|}{3}\right)$$

Otherwise

$$R_Z(\tau) = 10\left(\frac{3-|\tau|}{3}\right)$$

$$R_Z(\tau) = \begin{cases} 10 \ exp\left(-\frac{|\tau|}{3}\right) + 10\left(\frac{3-|\tau|}{3}\right) & 0 \le |\tau| \le 3\\ \\ 10\left(\frac{3-|\tau|}{3}\right) & otherwise \end{cases}$$

From the above discussions it is observed that

1. The mean of $R_Z(\tau) = 0$

2. $R_Z(\tau)$ is independent of absolute time, hence it is wide-sense stationary.

Total power or AC power of $R_Z(\tau)$ is

$$= \sigma_X^2 + \sigma_Y^2 = 10 + 10$$
$$= 20$$



23. The random process X(t) has the autocorrelation function

$$R_X(\tau) = \begin{cases} 10 \ exp(1 - |\tau|/\tau_N) & -\tau_N \le \tau \le \tau_N \\ 0 & otherwise \end{cases}$$

The random process Y(t) is independent of X(t) and has the autocorrelation function

$$R_Y(\tau) = 13 \left(\frac{\sin(\omega_{B\tau})}{\omega_{B\tau}} \right)$$

where $(2\pi)/\omega_B \gg \tau_N$. The random process Z(t) = X(t) + Y(t). For Z(t), find the autocorrelation function, its total power, its dc power, and its ac power. Is Z(t) wide-sense stationary? [?]

Solution:

It is mentioned that X(t) and Y(t) are jointly wide-sense stationary and they are independent. $\sigma_X^2 =$ $10, \mu_X = 0, \sigma_Y^2 = 13, \mu_Y = 0$, Autocorrelation function $R_Z(\tau)$

$$R_W(\tau) = R_X(\tau) + R_Y(\tau)$$

 $R_Z(\tau)$ is summation of the autocorrelation function of $R_X(\tau)$ and $R_Y(\tau)$, $R_X(\tau)$ is within the limits of $R_Y(\tau)$, hence we have to consider the limits for $R_Z(\tau)$ from $0 \le |\tau| \le \tau_N$ and other than this limits From $0 \leq |\tau| \leq \tau_N$ the $R_Z(\tau)$ is

$$R_Z(\tau) = 10 \exp(1 - |\tau|/\tau_N) + 13 \left(\frac{\sin(\omega_{B\tau})}{\omega_{B\tau}}\right)$$

Otherwise

$$R_Z(\tau) = 13 \left(\frac{\sin(\omega_{B\tau})}{\omega_{B\tau}} \right)$$

$$R_{Z}(\tau) = \begin{cases} 10(1 - |\tau|/\tau_{N}) + 13\left(\frac{\sin(\omega_{B\tau})}{\omega_{B\tau}}\right) & 0 \le |\tau| \le 3\\\\ 13\left(\frac{\sin(\omega_{B\tau})}{\omega_{B\tau}}\right) & otherwise \end{cases}$$

From the above discussions it is observed that

- 1. The mean of $R_Z(\tau) = 0$
- 2. $R_Z(\tau)$ is independent of absolute time, hence it is wide-sense stationary.

Total power or AC power of $R_Z(\tau)$ is

$$= \sigma_X^2 + \sigma_Y^2 = 10 + 13$$
$$= 23$$



24. A random process X(t) has the autocorrelation function

$$R_X(\tau) = 15exp(-2|\tau|) \quad -\infty < \tau < \infty$$

The random process Y(t) is Y(t) = X(t) - 3

- a. What is the autocorrelation function of Y(t)?
- b. What are the total power, the dc power, and the ac power of Y(t)?
- c. What is the cross-correlation $R_{XY}(\tau)$? [?]

Solution:

It is given that $R_X(\tau)$ has $\sigma_X^2 = 15$ and $\mu_X = 0$. It is given that $Y(t) = X(t) - 3 \sigma_Y^2 = 15$ and $\mu_X = -3$.

a. The autocorrelation function of Y(t) is

$$R_Y(\tau) = E[Y(t)Y(t+\tau)] = E[\{X(t) - 3\}\{X(t+\tau) - 3\}]$$

= $E[X(t)X(t+\tau) - 3X(t+\tau) - 3X(t) + 9]$
= $R_{XY}(\tau) + 9$

b. The total power, the dc power, and the ac power of Y(t) are The total power

$$E[Y^2(t)] = \sigma_Y^2 + \mu_Y^2 = 15 + 9 = 24$$

The DC power is

$$\mu_Y^2 = 9$$

The AC power is

$$\sigma_Y^2 = 15$$

c. The cross-correlation $R_{XY}(\tau)$ is

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)] = E[\{X(t)\}\{X(t+\tau) - 3\}]$$

= $E[X(t)X(t+\tau) - 3X(t)]$
= $R_X(\tau)$

25. A random process X(t) has the autocorrelation function

$$R_X(\tau) = 10\cos(\omega_c \tau) - \infty < \tau < \infty$$

The random process Y(t) is Y(t) = X(t) - 4

- a. What is the autocorrelation function of Y(t)?
- b. What are the total power, the dc power, and the ac power of Y(t)?
- c. What is the cross-correlation $R_{XY}(\tau)$? [?]



Solution:

It is given that $R_X(\tau)$ has $\sigma_X^2 = 10$ and $\mu_X = 0$. It is given that $Y(t) = X(t) - 4 \sigma_Y^2 = 10$ and $\mu_X = -4$.

a. The autocorrelation function of Y(t) is

$$R_Y(\tau) = E[Y(t)Y(t+\tau)] = E[\{X(t)-4\}\{X(t+\tau)-4\}]$$

= $E[X(t)X(t+\tau) - 4X(t+\tau) - 4X(t) + 16]$
= $R_{XY}(\tau) + 16$

b. The total power, the dc power, and the ac power of Y(t) are The total power

$$E[Y^2(t)] = \sigma_Y^2 + \mu_Y^2 = 10 + 16 = 26$$

The DC power is

$$\mu_Y^2 = 16$$

The AC power is

$$\sigma_{Y}^{2} = 10$$

c. The cross-correlation $R_{XY}(\tau)$ is

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)] = E[\{X(t)\}\{X(t+\tau)-4\}]$$

= $E[X(t)X(t+\tau) - 4X(t)]$
= $R_X(\tau)$

26. A random process X(t) has the autocorrelation function

$$R_X(\tau) = \begin{cases} 13(1 - |\tau|/T) & -T \le \tau \le T \\ 0 & otherwise \end{cases}$$

The random process Y(t) is Y(t) = X(t) - 2

- a. What is the autocorrelation function of Y(t)?
- b. What are the total power, the dc power, and the ac power of Y(t)?
- c. What is the cross-correlation $R_{XY}(\tau)$? [?]

Solution:

- It is given that $R_X(\tau)$ has $\sigma_X^2 = 13$ and $\mu_X = 0$. It is given that $Y(t) = X(t) 2 \sigma_Y^2 = 13$ and $\mu_X = -2$.
- a. The autocorrelation function of Y(t) is

$$R_Y(\tau) = E[Y(t)Y(t+\tau)] = E[\{X(t)-2\}\{X(t+\tau)-2\}]$$

= $E[X(t)X(t+\tau)-2X(t+\tau)-2X(t)+4]$
= $R_{XY}(\tau)+4$

b. The total power, the dc power, and the ac power of Y(t) are



The total power

$$E[Y^2(t)] = \sigma_Y^2 + \mu_Y^2 = 13 + 4 = 17$$

The DC power is

 $\mu_Y^2 = 4$

The AC power is

 $\sigma_Y^2 = 13$

c. The cross-correlation $R_{XY}(\tau)$ is

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)] = E[\{X(t)\}\{X(t+\tau)-2\}]$$

= $E[X(t)X(t+\tau)-2X(t)]$
= $R_X(\tau)$

27. X(t) and Y(t) are zero mean jointly wide-sense stationary random process. The random process Z(t) is

$$Z(t) = 3X(t) + Y(t)$$

Find the correlations $R_Z(\tau), R_{ZX}(\tau), R_{XZ}(\tau), R_{ZY}(\tau)$ and $R_{YZ}(\tau)$ [?]

Solution:

$$\begin{split} R_{Z}(\tau) &= E[Z(t)Z(t+\tau)] = E[\{3X(t) + Y(t)\}\{3X(t+\tau) + Y(t+\tau)\}] \\ &= E[\{9X(t)X(t+\tau)\} + \{3X(t)Y(t+\tau)\} + \{3X(t+\tau)Y(t)\} + \{Y(t)Y(t+\tau)\}] \\ &= 9R_{X}(\tau) + 3R_{XY}(\tau) + 3R_{YX}(\tau) + R_{Y}(\tau) \\ R_{ZX}(\tau) &= E[Z(t)X(t+\tau)] = E[\{3X(t) + Y(t)\}X(t+\tau)] \\ &= E[\{3X(t)X(t+\tau)\} + \{Y(t)X(t+\tau)\}] \\ &= 3R_{X}(\tau) + R_{YX}(\tau) \\ R_{XZ}(\tau) &= E[X(t)Z(t+\tau)] = E[X(t)\{3X(t+\tau) + Y(t+\tau)\}] \\ &= 3R_{X}(\tau) + R_{XY}(\tau) \\ R_{ZY}(\tau) &= E[Z(t)Y(t+\tau)] = E[\{3X(t) + Y(t)\}Y(t+\tau)] \\ &= E[\{3X(t)Y(t+\tau)\} + \{Y(t)Y(t+\tau)\}] \\ &= 3R_{XY}(\tau) + R_{Y}(\tau) \\ R_{YZ}(\tau) &= E[Y(t)Z(t+\tau)] = E[Y(t)\{3X(t+\tau) + Y(t+\tau)\}] \\ &= E[\{3Y(t)X(t+\tau)\} + \{Y(t)Y(t+\tau)\}] \\ &= E[\{3Y(t)X(t+\tau)\} + \{Y(t)Y(t+\tau)\}] \\ &= 3R_{YX}(\tau) + R_{Y}(\tau) \end{split}$$

28. X(t) and Y(t) are zero mean jointly wide-sense stationary random process. The random process Z(t) is

$$Z(t) = 3X(t) + 2Y(t)$$

Find the correlations $R_Z(\tau), R_{ZX}(\tau), R_{XZ}(\tau), R_{ZY}(\tau)$ and $R_{YZ}(\tau)$ [?]

Solution:

$$\begin{aligned} R_{Z}(\tau) &= E[Z(t)Z(t+\tau)] = E[\{3X(t) + 2Y(t)\}\{3X(t+\tau) + 2Y(t+\tau)\}] \\ &= E[\{9X(t)X(t+\tau)\} + \{6X(t)Y(t+\tau)\} + \{6X(t+\tau)Y(t)\} + 4\{Y(t)Y(t+\tau)\}] \\ &= 9R_{X}(\tau) + 6R_{XY}(\tau) + 6R_{YX}(\tau) + 4R_{Y}(\tau) \end{aligned}$$
$$\begin{aligned} R_{ZX}(\tau) &= E[Z(t)X(t+\tau)] = E[\{3X(t) + 2Y(t)\}X(t+\tau)] \\ &= E[\{3X(t)X(t+\tau)\} + 2\{Y(t)X(t+\tau)\}] \\ &= 3R_{X}(\tau) + 2R_{YX}(\tau) \end{aligned}$$
$$\begin{aligned} R_{XZ}(\tau) &= E[X(t)Z(t+\tau)] = E[X(t)\{3X(t+\tau) + 2Y(t+\tau)\}] \\ &= E[\{3X(t)X(t+\tau)\} + 2\{X(t)Y(t+\tau)\}] \\ &= 3R_{X}(\tau) + 2R_{XY}(\tau) \end{aligned}$$
$$\begin{aligned} R_{ZY}(\tau) &= E[Z(t)Y(t+\tau)] = E[\{3X(t) + 2Y(t)\}Y(t+\tau)] \\ &= E[\{3X(t)Y(t+\tau)\} + 2\{Y(t)Y(t+\tau)\}] \\ &= 3R_{XY}(\tau) + 2R_{YY}(\tau) \end{aligned}$$

$$R_{YZ}(\tau) = E[Y(t)Z(t+\tau)] = E[Y(t)\{3X(t+\tau)+2Y(t+\tau)\}]$$

= $E[\{6Y(t)X(t+\tau)\}+2\{Y(t)Y(t+\tau)\}]$
= $6R_{YX}(\tau)+2R_Y(\tau)$

29. X(t) and Y(t) are zero mean jointly wide-sense stationary random process. The random process Z(t) is

$$Z(t) = X(t) + 2Y(t)$$

Find the correlations $R_Z(\tau), R_{ZX}(\tau), R_{XZ}(\tau), R_{ZY}(\tau)$ and $R_{YZ}(\tau)$ [?]

Solution:

$$\begin{aligned} R_Z(\tau) &= E[Z(t)Z(t+\tau)] = E[\{X(t) + 2Y(t)\}\{X(t+\tau) + 2Y(t+\tau)\}] \\ &= E[\{X(t)X(t+\tau)\} + \{2X(t)Y(t+\tau)\} + \{2X(t+\tau)Y(t)\} + 4\{Y(t)Y(t+\tau)\}] \\ &= R_X(\tau) + 2R_{XY}(\tau) + 2R_{YX}(\tau) + 4R_Y(\tau) \end{aligned}$$
$$\begin{aligned} R_{ZX}(\tau) &= E[Z(t)X(t+\tau)] = E[\{X(t) + 2Y(t)\}X(t+\tau)] \\ &= E[\{X(t)X(t+\tau)\} + 2\{Y(t)X(t+\tau)\}] \\ &= R_X(\tau) + 2R_{YX}(\tau) \end{aligned}$$



$$R_{XZ}(\tau) = E[X(t)Z(t+\tau)] = E[X(t)\{X(t+\tau) + 2Y(t+\tau)\}]$$

$$= E[\{X(t)X(t+\tau)\} + 2\{X(t)Y(t+\tau)\}]$$

$$= R_X(\tau) + 2R_{XY}(\tau)$$

$$R_{ZY}(\tau) = E[Z(t)Y(t+\tau)] = E[\{X(t) + 2Y(t)\}Y(t+\tau)]$$

$$= E[\{X(t)Y(t+\tau)\} + 2\{Y(t)Y(t+\tau)\}]$$

$$= R_{XY}(\tau) + 2R_Y(\tau)$$

$$R_{YZ}(\tau) = E[Y(t)Z(t+\tau)] = E[Y(t)\{X(t+\tau) + 2Y(t+\tau)\}]$$

$$= E[\{Y(t)X(t+\tau)\} + 2\{Y(t)Y(t+\tau)\}]$$

$$= R_{YX}(\tau) + 2R_Y(\tau)$$

Multiplication of Random Process: 1.4.4

Consider the multiplication of two random independent, jointly wide-sense stationary random precess X(t)and Y(t) is

$$W(t) = X(t)Y(t)$$

The mean of W(t) is

$$E[W(t)] = \mu_W = E[X(t)]E[Y(t)] = \mu_X \mu_Y$$

The variance of W(t) is

$$\sigma_W^2 = E[X(t)]E[Y(t) - \mu_X \mu_Y]^2 = (\sigma_X^2 + \mu_X^2)(\sigma_Y^2 + \mu_Y^2)\mu_X^2 \mu_Y^2$$

The autocorrelation function of the sum W(t) is

$$R_W(\tau) = E[\{X(t)Y(t)\}\{X(t+\tau) + Y(t+\tau)\}]$$

= $E[X(t)X(t+\tau)]E[Y(t)Y(t+\tau)]$
= $R_X(\tau)R_Y(\tau)$

30. The random process X(t) is noise with the autocorrelation function

$$R_X(\tau) = \begin{cases} 10(1 - |\tau|/\tau_N) & -\tau_N \le \tau \le \tau_N \\ 0 & otherwise \end{cases}$$

The random process Y(t) is independent of X(t) and has the autocorrelation function

$$R_Y(\tau) = \begin{cases} 15(1 - |\tau|/T) & -T \le t \le T\\ 0 & otherwise \end{cases}$$

where $T \gg \tau_N$. The random process $Z(t) = X(t) \times Y(t)$. For Z(t), find the autocorrelation function, its total power, its dc power, and its ac power. Is Z(t) wide-sense stationary? [?]

Solution:

 $\sigma_X^2 = 10, \mu_X = 0, \sigma_Y^2 = 15, \mu_Y = 0,$ Autocorrelation function $R_Z(\tau)$

$$R_Z(\tau) = R_X(\tau) \times R_Y(\tau)$$



$$R_Z(\tau) = 15(1 - |\tau|/T) \times 10(1 - |\tau|/\tau_N)$$

= 150(1 - |\tau|/T)(1 - |\tau|/\tau_N)

$$R_{Z}(\tau) = \begin{cases} 150(1 - |\tau|/T)(1 - |\tau|/\tau_{N}) & |\tau| \le \tau_{N} \\ 0 & otherwise \end{cases}$$

From the above discussions it is observed that

- 1. The mean of $R_Z(\tau) = 0$
- 2. $R_Z(\tau)$ is independent of absolute time, hence it is wide-sense stationary.

Total power or AC power of $R_Z(\tau)$ is 150

31. The random process X(t) is noise with the autocorrelation function

$$R_X(\tau) = \begin{cases} 10(1 - |\tau|/\tau_N) & -\tau_N \le \tau \le \tau_N \\ 0 & otherwise \end{cases}$$

The random process Y(t) is independent of X(t) and has the autocorrelation function

$$R_Y(\tau) = 18\cos(\omega_c \tau) \quad -\infty \le \tau \le \infty$$

where $2\pi/\omega_c \gg \tau_N$. The random process $Z(t) = X(t) \times Y(t)$. For Z(t), find the autocorrelation function, its total power, its dc power, and its ac power. Is Z(t) wide-sense stationary? [?]

Solution:

 $\sigma_X^2 = 10, \mu_X = 0, \sigma_Y^2 = 18, \mu_Y = 0,$ Autocorrelation function $R_Z(\tau)$

$$R_Z(\tau) = R_X(\tau) \times R_Y(\tau)$$

$$R_Z(\tau) = 15(1 - |\tau|/T) \times 18\cos(\omega_c \tau)$$

= 180(1 - |\tau|/T)(\cos(\omega_c \tau))

$$R_Z(\tau) = \begin{cases} 180(\cos(\omega_c \tau))(1 - |\tau|/T) & |\tau| \le \tau_N \\ 0 & otherwise \end{cases}$$

From the above discussions it is observed that

- 1. The mean of $R_Z(\tau) = 0$
- 2. $R_Z(\tau)$ is independent of absolute time, hence it is wide-sense stationary.

Total power or AC power of $R_Z(\tau)$ is 180

32. The random process X(t) is noise with the autocorrelation function

$$R_X(\tau) = \begin{cases} 10(1 - |\tau|/\tau_N) & -\tau_N \le \tau \le \tau_N \\ 0 & otherwise \end{cases}$$



The random process Y(t) is independent of X(t) and has the autocorrelation function

$$R_Y(\tau) = 13 \frac{\sin(\omega_B \tau)}{(\omega_B \tau)} \quad -\infty \le \tau \le \infty$$

where $2\pi/\omega_B \gg \tau_N$. The random process $Z(t) = X(t) \times Y(t)$. For Z(t), find the autocorrelation function, its total power, its dc power, and its ac power. Is Z(t) wide-sense stationary? [?]

Solution:

 $\sigma_X^2 = 10, \mu_X = 0, \sigma_Y^2 = 13, \mu_Y = 0$, Autocorrelation function $R_Z(\tau)$

$$R_Z(\tau) = R_X(\tau) \times R_Y(\tau)$$

$$R_Z(\tau) = 10(1 - |\tau|/T) \times 13 \frac{\sin(\omega_B \tau)}{(\omega_B \tau)}$$
$$= 130(1 - |\tau|/T) \frac{\sin(\omega_B \tau)}{(\omega_B \tau)}$$

$$R_Z(\tau) = \begin{cases} 130 \frac{\sin(\omega_B \tau)}{(\omega_B \tau)} (1 - |\tau|/T) & |\tau| \le \tau_N \\ 0 & otherwise \end{cases}$$

From the above discussions it is observed that

- 1. The mean of $R_Z(\tau) = 0$
- 2. $R_Z(\tau)$ is independent of absolute time, hence it is wide-sense stationary.

Total power or AC power of $R_Z(\tau)$ is 130

33. The random process X(t) and Y(t) are jointly wide-sense stationary and they are independent. Given that $W(t) = X(t) \times Y(t)$ and

$$R_X(\tau) = 10exp\left(-\frac{|\tau|}{3}\right) \quad -\infty \le \tau \le \infty$$
$$R_Y(\tau) = \begin{cases} 11\left(\frac{3-|\tau|}{3}\right) & -3 \le \tau \le 3\\ 0 & otherwise \end{cases}$$

For W(t) find its autocorrelation function, its total power, its dc power, and its ac power. Is W(t) wide-sense stationary? |?|

Solution:

 $\sigma_X^2 = 10, \mu_X = 0, \sigma_Y^2 = 11, \mu_Y = 0,$ Autocorrelation function $R_W(\tau)$

$$R_W(\tau) = R_X(\tau) \times R_Y(\tau)$$

$$R_W(\tau) = 10exp\left(-\frac{|\tau|}{3}\right) \times 11\left(\frac{3-|\tau|}{3}\right)$$
$$= 110exp\left(-\frac{|\tau|}{3}\right)\left(\frac{3-|\tau|}{3}\right)$$



$$R_W(\tau) = \begin{cases} 110exp\left(-\frac{|\tau|}{3}\right)\left(\frac{3-|\tau|}{3}\right) & |\tau| \le \tau_N\\ 0 & otherwise \end{cases}$$

From the above discussions it is observed that

- 1. The mean of $R_W(\tau) = 0$
- 2. $R_W(\tau)$ is independent of absolute time, hence it is wide-sense stationary.

Total power or AC power of $R_W(\tau)$ is 110

The random process X(t) and Y(t) are jointly wide-sense stationary and they are **34**. independent. Given that $W(t) = X(t) \times Y(t)$ and

$$R_X(\tau) = 12exp\left(-\frac{|\tau|}{4}\right) \quad -\infty \le \tau \le \infty$$
$$R_Y(\tau) = \begin{cases} 10\left(\frac{4-|\tau|}{4}\right) & -4 \le \tau \le 4\\ 0 & otherwise \end{cases}$$

For W(t) find its autocorrelation function, its total power, its dc power, and its ac power. Is W(t) wide-sense stationary? [?]

Solution:

 $\sigma_X^2 = 12, \mu_X = 0, \sigma_Y^2 = 10, \mu_Y = 0,$ Autocorrelation function $R_W(\tau)$

$$R_W(\tau) = R_X(\tau) \times R_Y(\tau)$$

$$R_W(\tau) = 12exp\left(-\frac{|\tau|}{4}\right) \times 10\left(\frac{4-|\tau|}{4}\right)$$
$$= 120exp\left(-\frac{|\tau|}{4}\right)\left(\frac{4-|\tau|}{4}\right)$$

$$R_W(\tau) = \begin{cases} 120exp\left(-\frac{|\tau|}{4}\right)\left(\frac{4-|\tau|}{4}\right) & |\tau| \le 4\\ 0 & otherwise \end{cases}$$

From the above discussions it is observed that

- 1. The mean of $R_W(\tau) = 0$
- 2. $R_W(\tau)$ is independent of absolute time, hence it is wide-sense stationary.

Total power or AC power of $R_W(\tau)$ is 120

35. The random process X(t) and Y(t) are jointly wide-sense stationary and they are independent. Given that $W(t) = X(t) \times Y(t)$ and

$$R_X(\tau) = 11exp\left(-\frac{|\tau|}{5}\right) \quad -\infty \le \tau \le \infty$$



$$R_Y(\tau) = \begin{cases} 12\left(\frac{5-|\tau|}{5}\right) & -5 \le \tau \le 5\\ 0 & otherwise \end{cases}$$

For W(t) find its autocorrelation function, its total power, its dc power, and its ac power. Is W(t) wide-sense stationary? [?]

Solution:

 $\sigma_X^2 = 11, \mu_X = 0, \sigma_Y^2 = 12, \mu_Y = 0,$ Autocorrelation function $R_W(\tau)$

$$R_W(\tau) = R_X(\tau) \times R_Y(\tau)$$

$$R_W(\tau) = 11exp\left(-\frac{|\tau|}{5}\right) \times 12\left(\frac{5-|\tau|}{5}\right)$$
$$= 132exp\left(-\frac{|\tau|}{5}\right)\left(\frac{5-|\tau|}{5}\right)$$

$$R_W(\tau) = \begin{cases} 132exp\left(-\frac{|\tau|}{5}\right)\left(\frac{5-|\tau|}{5}\right) & |\tau| \le 5\\ 0 & otherwise \end{cases}$$

From the above discussions it is observed that

- 1. The mean of $R_W(\tau) = 0$
- 2. $R_W(\tau)$ is independent of absolute time, hence it is wide-sense stationary.

Total power or AC power of $R_W(\tau)$ is 132



1.5**Ergodic Random Processes**

Any random process is wide-sense stationarity if it satisfies the conditions like mean and and autocorrelation function are independent of time. To calculate mean and autocorrelation function of a random process, it requires an ensemble of sample functions (data records). It is difficult to collect the data in real time situations. In many real-life applications Its convenient to calculate the averages from a single data record. This is possible in certain random processes called ergodic processes.

A random process is ergodic if time averages with a sample function equal to ensemble. The avaraging of random process is defined as

$$\langle x(t) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) dt$$

The autocorrelation function for random process x(t) is defined as

$$\langle x(t)x(t+\tau)\rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t+\tau)dt$$

36. Assume that the data in the following table are obtained from a windowed sample function obtained from an ergodic random process. Use(4.73) to estimate the autocorrelation for $\tau = 0$, 2 and 4 ms, where $\Delta t = 2ms$

$\mathbf{x}(t)$	1.5	2.1	1.0	2.2	-1.6	-2.0	-2.5	2.5	1.6	-1.8
k	0	1	2	3	4	5	6	7	8	9

Solution:

Autocorrelation function for discrete sequences is

$$r_X(i) = \frac{1}{n} \sum_{k=0}^{k=N-1-i} x(k)x(k+i)$$

It is given that $\tau = 0$, 2 and 4 ms, where $\Delta t = 2ms$, when $\tau = 0$ it is an autocorrelation when $\tau = 2$ and $\Delta t = 2ms$ i = 2/2 = 1, $\tau = 4$ and $\Delta t = 2ms$ i = 4/2 = 2

$$\begin{aligned} r_X(i) &= \frac{1}{n} \sum_{k=0}^{k=N-1-i} x(k)x(k+i) \\ r_X(0) &= \frac{1}{n} \sum_{k=0}^{k=N-1} x(k)x(k) = \frac{1}{n} \left[x(k)^2 \right] \\ &= \frac{1}{10} \left[(1.5)^2 + (2.1)^2 + (1.0)^2 + (2.2)^2 + (-1.6)^2 + (-2.0)^2 + (-2.5)^2 + (2.5)^2 + (1.6)^2 + (-1.8)^2 \right] \\ &= 3.736 \\ r_X(2\ ms) &= \frac{1}{n} \sum_{k=0}^{k=N-2} x(k)x(k+1) \\ &= \frac{1}{10} [(1.5)(2.1) + (2.1)(1.0) + (1.0)(2.2) + (2.2)(-1.6) + (-1.6)(-2.0) + (-2.0)(-2.5) \\ &+ (-2.5)(2.5) + (2.5)(1.6) + (1.6)(-1.8)] \\ &= \frac{1}{10} [3.15 + 2.1 + 2.2 - 3.52 + 3.2 + 5 - 6.25 + 4 - 2.88] \\ &= -0.700 \end{aligned}$$



$$r_X(4 \ ms) = \frac{1}{n} \sum_{k=0}^{k=N-3} x(k)x(k+2)$$

= $\frac{1}{10} [(1.5)(1.0) + (2.1)(2.2) + (1.0)(-1.6) + (2.2)(-2.0) + (-1.6)(-2.5) + (-2.0)(2.5) + (-2.5)(1.6) + (2.5)(-1.8)]$
= -0.938

37. Assume that the data in the following table are obtained from a windowed sample function obtained from an ergodic random process. Use(4.73) to estimate the autocorrelation for $\tau = 0$, 3 and 6 ms, where $\Delta t = 3ms$

$\mathbf{x}(t)$	1.0	2.2	1.5	-3.0	-0.5	1.7	-3.5	-1.5	1.6	-1.3
k	0	1	2	3	4	5	6	7	8	9

Solution:

Autocorrelation function for discrete sequences is

$$r_X(i) = \frac{1}{n} \sum_{k=0}^{k=N-1-i} x(k)x(k+i)$$

It is given that $\tau = 0, \ 3 \ and \ 6 \ ms$, where $\Delta t = 3ms$, when $\tau = 0$ it is an autocorrelation when $\tau = 3$ and $\Delta t = 3ms$ i = 3/3 = 1, $\tau = 6$ and $\Delta t = 23ms$ i = 6/3 = 2

$$r_X(i) = \frac{1}{n} \sum_{k=0}^{k=N-1-i} x(k)x(k+i)$$

$$r_X(0) = \frac{1}{n} \sum_{k=0}^{k=N-1} x(k)x(k) = \frac{1}{n} [x(k)^2]$$

$$= \frac{1}{10} [(1.0)^2 + (2.2)^2 + (1.5)^2 + (-3.0)^2 + (-0.5)^2 + (1.7)^2 + (-3.5)^2 + (-1.5)^2 + (1.6)^2 + (-1.3)^2]$$

$$= 3.898$$

$$r_X(3 \ ms) = \frac{1}{n} \sum_{k=0}^{k=N-2} x(k)x(k+1)$$

= $\frac{1}{10} [(1.0)(2.2) + (2.2)(1.5) + (1.5)(-3.0) + (-3.0)(-0.5) + (-0.5)(1.7) + (1.7)(-3.5) + (-3.5)(-1.5) + (-1.5)(1.6) + (1.6)(-1.3)]$
= -0.353

$$r_X(6 \ ms) = \frac{1}{n} \sum_{k=0}^{k=N-3} x(k)x(k+2)$$

= $\frac{1}{10} [(1.0)(1.5) + (2.2)(-3.0) + (1.5)(-0.5) + (-3.0)(1.7) + (-0.5)(-3.5) + (1.7)(-1.5) + (-3.5)(1.6) + (-1.5)(-1.3)]$
= -1.540



38. Assume that the data in the following table are obtained from a windowed sample function obtained from an ergodic random process. Use(4.73) to estimate the autocorrelation for $\tau = 0$, 7 and 14 ms, where $\Delta t = 7ms$

$\mathbf{x}(t)$	1.5	0.4	0.8	0.3	-0.4	-1.7	2.0	-2.0	0.8	-0.2
k	0	1	2	3	4	5	6	7	8	9

Solution:

Autocorrelation function for discrete sequences is

$$r_X(i) = \frac{1}{n} \sum_{k=0}^{k=N-1-i} x(k)x(k+i)$$

It is given that $\tau = 0, 2 \text{ and } 4 \text{ ms}$, where $\Delta t = 2ms$, when $\tau = 0$ it is an autocorrelation when $\tau = 7$ and $\Delta t = 7ms$ i = 7/7 = 1, $\tau = 14$ and $\Delta t = 7ms$ i = 14/7 = 2

$$r_X(i) = \frac{1}{n} \sum_{k=0}^{k=N-1-i} x(k)x(k+i)$$

$$r_X(0) = \frac{1}{n} \sum_{k=0}^{k=N-1} x(k)x(k) = \frac{1}{n} [x(k)^2]$$

$$= \frac{1}{10} [(1.5)^2 + (0.4)^2 + (0.8)^2 + (0.3)^2 + (-0.4)^2 + (-1.7)^2 + (2.0)^2 + (-2.0)^2 + (0.8)^2 + (-0.2)^2]$$

$$= 1.487$$

$$\begin{aligned} r_X(7\ ms) &= \frac{1}{n} \sum_{k=0}^{k=N-1} x(k)x(k+1) \\ &= \frac{1}{10} [(1.5)(0.4) + (0.4)(0.8) + (0.8)(0.3) + (0.3)(-0.4) + (-0.4)(-1.7) + (-1.7)(2.0) \\ &+ (2.0)(-2.0) + (-2.0)(0.8) + (0.8)(-0.2)] \\ &= -0.744 \\ r_X(14\ ms) &= \frac{1}{n} \sum_{k=0}^{k=N-2} x(k)x(k+2) \\ &= \frac{1}{10} [(1.5)(0.8) + (0.4)(0.3) + (0.8)(-0.4) + (0.3)(-1.7) + (-0.4)(2.0) + (-1.7)(-2.0) \\ &+ (2.0)(0.8) + (-2.0)(-0.2)] \\ &= 0.509 \end{aligned}$$



Power Spectral Densities 1.6

The power spectral density of a wide-sense stationary random process is the Fourier transform of the autocorrelation function. The distribution of power over range frequencies is described by power power density spectrum.

Fourier transform pair is

$$X_T(j\omega) = \int_{-\infty}^{\infty} x_T(t)e^{-j\omega t}dt$$
$$x_T(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_T(j\omega)e^{j\omega t}d\omega$$

The power of signal is its square. Averaging the power of the signal x(t) within a window of width T is

$$P_T = \frac{1}{T} \int_{-T/2}^{T/2} x^2(t) dt$$

$$= \frac{1}{T} \int_{-\infty}^{\infty} x_T^2(t) dt$$

$$= \frac{1}{T} \int_{-\infty}^{\infty} x_T(t) \frac{1}{2\pi} \int_{-\infty}^{\infty} x_T(t) e^{j\omega} d\omega dt$$

$$= \frac{1}{2\pi T} \int_{-\infty}^{\infty} x_T(j\omega) \frac{1}{2\pi} \int_{-\infty}^{\infty} x_T(t) e^{j\omega t} dt d\omega$$

$$= \frac{1}{2\pi T} \int_{-\infty}^{\infty} x_T(j\omega) x_T(-j\omega) d\omega$$

$$x_T(-j\omega) = X_T^*(j\omega)$$

$$P_T = \frac{1}{T} \int_{-T/2}^{T/2} X^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_T(j\omega)|^2 d\omega$$
$$\int_{-T/2}^{T/2} X^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_T(j\omega)|^2 d\omega$$

Expectation of the above equation is

$$E\left[\int_{-T/2}^{T/2} X^{2}(t)dt\right] = \frac{1}{2\pi}E\left[\int_{-\infty}^{\infty} |X_{T}(j\omega)|^{2}\right]d\omega$$

$$\int_{-T/2}^{T/2} E[X^{2}(t)]dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} E[|X_{T}(j\omega)|^{2}]d\omega$$

$$TE[X^{2}(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} E[|X_{T}(j\omega)|^{2}]d\omega$$

$$TR_{X}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E[|X_{T}(j\omega)|^{2}]d\omega$$

$$R_{X}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T \to \infty} E[|X_{T}(j\omega)|^{2}]d\omega$$

The above equation can be split into two pars as

$$R_X(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) d\omega$$

$$S_X(\omega) = \lim_{T \to \infty} E[|X_T(j\omega)|^2]$$



39. A PSD is as shown in Figure 1.2 where the constants are a = 55, b = 5, $\omega_o = 1000$ and $\omega_1 = 100$. Calculate values for $E[X^2(t)]$, the σ_X^2 and $|\mu_X|$.



Figure 1.2

Solution:



Figure 1.3

 $\omega_o - \omega_1 = 1000 - 100 = 900$ $\omega_o + \omega_1 = 1000 + 100 = 1100$ $(\omega_o + \omega_1) - (\omega_o - \omega_1) = 1100 - 1000 = 200$

The given spectrum is in the triangular form, its base is 200 and its height is 5 and it also has a = 55. The modified two sided spectrum is redrawn and is as shown in Figure 1.3. The x axis is ω radians/sec. The Autocorrelation function is

$$R_X(0) = E[X^2(t)] = Area \ under \ PSD = Area \ of \ impulse + Area \ of \ Triangle$$
$$= 2 \times \frac{1}{2\pi} [Area \ of \ impulse + Area \ of \ Triangle]$$
$$= \sigma_X^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) d\omega$$
$$= \frac{2}{2\pi} \left(55 + \frac{1}{2}(5)(200) \right)$$
$$= 176.6620$$
$$|\mu_X| = 0$$

40. A PSD is as shown in Figure 1.2 where the constants are $a = 450, b = 6, \omega_o = 10,000$ and

 $\omega_1 = 1000$. Calculate values for $E[X^2(t)]$, the σ_X^2 and $|\mu_X|$.

Solution:

The given spectrum has the following details constant a = 450

$$\begin{aligned} \omega_o - \omega_1 &= 10,000 - 1000 = 9900\\ \omega_o + \omega_1 &= 10000 + 1000 = 11000\\ (\omega_o + \omega_1) - (\omega_o - \omega_1) &= 11000 - 9000 = 2000 \end{aligned}$$

The signal magnitude is b = 6



$$R_X(0) = E[X^2(t(] = \sigma_X^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) d\omega]$$

= $\frac{2}{2\pi} \left(450 + \frac{1}{2}(6)(2000) \right)$
= 2053.0988
 $|\mu_X| = 0$

41. A PSD is as shown in Figure 1.2 where the constants are a = 72, b = 4, $\omega_o = 1000$ and $\omega_1 = 50$. Calculate values for $\widetilde{E}[X^2(t)]$, the σ_X^2 and $|\mu_X|$.

Solution:

The given spectrum has the following details constant a = 72

$$\begin{aligned} \omega_o - \omega_1 &= 1000 - 50 = 950 \\ \omega_o + \omega_1 &= 1000 + 50 = 1050 \\ (\omega_o + \omega_1) - (\omega_o - \omega_1) &= 1050 - 950 = 100 \end{aligned}$$

The signal magnitude is b = 4

Autocorrelation function is

$$R_X(0) = E[X^2(t()] = \sigma_X^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) d\omega$$

= $\frac{2}{2\pi} \left(72 + \frac{1}{2} (4) (100) \right)$
= 86.5803
 $|\mu_X| = 0$

42. A PSD is as shown in Figure 1.4 where the constants are a = 3, b = 5, $\omega_1 = 8$ and $\omega_2 = 12$. Calculate values for $R_X(0)$, the variance and μ_X^2 .



Figure 1.4

Solution:

 $\omega_1 =$ 8 $\omega_2 = 12$ $\omega_2 - \omega_1 = 12 - 8 = 4$



The given spectrum is in the rectangular form, its base is 4 and its height is 5 and it also has a = 3 at $\omega = 0$. The x axis is ω radians/sec. The Autocorrelation function is

$$R_X(0) = E[X^2(t)] = Area \ under \ PSD = Area \ of \ impulse + Area \ of \ Rectangle = \frac{1}{2\pi} [Area \ of \ impulse + Area \ of \ Rectangle] = \mu_X^2 + \sigma_X^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) d\omega = \frac{1}{2\pi} (3 + 2[(5)(12 - 8)]) = 6.8437 \mu_X^2 = \frac{3}{2\pi} = 0.4775 \sigma_X^2 = R_X(0) - \mu_X^2 = 6.8437 - 0.4775 = 6.3662$$

43. A PSD is as shown in Figure 1.4 where the constants are a = 5, b = 3, $\omega_1 = 7$ and $\omega_2 = 13$. Calculate values for $R_X(0)$, the variance and μ_X^2 .

Solution:

The given spectrum has the following details constant a = 5

$$\begin{array}{rcl}
\omega_1 &=& 7\\
\omega_2 &=& 13\\
\omega_2 - \omega_1 &=& 13 - 7 = 6
\end{array}$$

The signal magnitude is b = 3

Autocorrelation function is

$$R_X(0) = \mu_X^2 + \sigma_X^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) d\omega$$

= $\frac{1}{2\pi} (5 + 2(3)(13 - 7))$
= 6.5254
 $\mu_X^2 = \frac{3}{2\pi} = 0.7958$
 $\sigma_X^2 = R_X(0) - \mu_X^2 = 6.5254 - 0.7958$
= 5.7296

44. A PSD is as shown in Figure 1.4 where the constants are a = 4, b = 3, $\omega_1 = 9$ and $\omega_2 = 14$. Calculate values for $R_X(0)$, the variance and μ_X^2 .

Solution:

The given spectrum has the following details constant a = 4

$$\begin{aligned}
\omega_1 &= 9\\ \omega_2 &= 14\\ \omega_2 - \omega_1 &= 14 - 9 = 5
\end{aligned}$$

The signal magnitude is b = 3



$$R_X(0) = \mu_X^2 + \sigma_X^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) d\omega$$

= $\frac{1}{2\pi} (4 + 2(3)(14 - 9))$
= 5.4113
 $\mu_X^2 = \frac{3}{2\pi} = 0.6366$
 $\sigma_X^2 = R_X(0) - \mu_X^2 = 5.4113 - 0.6366$
= 4.7746

45. A PSD is as shown in Figure 1.5 where the constants are a = 5, $\omega_o = 100$ and w = 8. Calculate values for $R_X(0)$, the variance and $|\mu_X|$.



Figure 1.5

Solution:

The given spectrum has the following details

$$w = 8$$

$$2w = 16$$

The signal magnitude is a = 5

Autocorrelation function is

$$R_X(0) = \sigma_X^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) d\omega$$
$$= \frac{2}{2\pi} (5(16))$$
$$= 25.4648$$
$$\mu_X = 0$$

46. A PSD is as shown in Figure 1.5 where the constants are a = 3, $\omega_o = 150$ and w = 7. Calculate values for $R_X(0)$, the variance and $|\mu_X|$.

Solution:

The given spectrum has the following details

w = 72w = 14

The signal magnitude is a = 3



$$R_X(0) = \sigma_X^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) d\omega$$
$$= \frac{2}{2\pi} (3(14))$$
$$= 13.3690$$
$$\mu_X = 0$$

47. A PSD is as shown in Figure 1.6 where the constants are a = 4, $\omega_o = 125$ and w = 6. Calculate values for $R_X(0)$, the variance and $|\mu_X|$.

Solution:

The given spectrum has the following details

$$w = 6$$
$$2w = 12$$

The signal magnitude is a = 3Autocorrelation function is

$$R_X(0) = \sigma_X^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) d\omega$$
$$= \frac{2}{2\pi} (4(12))$$
$$= 15.2789$$
$$\mu_X = 0$$

48. A PSD is as shown in Figure 1.6 where the constants are $a = 300, b = 10, \omega_M = 100$. Calculate values for $R_X(0)$, the variance and $|\mu_X|$.



Figure 1.6

Solution:

The given spectrum has the following details

$$-\omega_M + \omega_M = 100 + 100 = 200$$

The signal magnitude is a = 300, b = 10



$$\begin{aligned} R_X(0) &= \mu_X^2 + \sigma_X^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) d\omega \\ &= \frac{1}{2\pi} \left(300 + (10)(200) \right) \\ \sigma_X^2 &= \frac{2000}{2\pi} = 318.3099 \\ \mu_X^2 &= \frac{300}{2\pi} = 47.7465 \\ |\mu_X| &= 6.9099 \\ \sigma_X^2 &= R_X(0) - \sigma_X^2 = 318.3099 - 47.7465 = 366.0564 \end{aligned}$$

49. A PSD is as shown in Figure 1.6 where the constants are $a = 200, b = 20, \omega_M = 80$. Calculate values for $R_X(0)$, the variance and $|\mu_X|$.

Solution:

The given spectrum has the following details

$$-\omega_M + \omega_M = 80 + 80 = 160$$

The signal magnitude is a = 200, b = 20

Autocorrelation function is

$$R_X(0) = \mu_X^2 + \sigma_X^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) d\omega$$

= $\frac{1}{2\pi} (200 + (20)(160))$
= 541.1268
 $\mu_X^2 = \frac{300}{2\pi} = 47.7465$
 $|\mu_X| = 6.9099$
 $\sigma_X^2 = R_X(0) - \mu_X^2 = 541.1268 - 31.8310 = 509.2958$

50. A PSD is as shown in Figure 1.6 where the constants are $a = 300, b = 15, \omega_M = 75$. Calculate values for $R_X(0)$, the variance and $|\mu_X|$.

Solution:

The given spectrum has the following details

$$-\omega_M + \omega_M = 75 + 75 = 150$$

The signal magnitude is a = 300, b = 15

Autocorrelation function is

$$R_X(0) = \mu_X^2 + \sigma_X^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) d\omega$$

= $\frac{1}{2\pi} (300 + (15)(150))$
= 405.8451
 $\mu_X^2 = \frac{300}{2\pi} = 47.7465$
 $|\mu_X| = 6.9099$
 $\sigma_X^2 = R_X(0) - \mu_X^2 = 405.8451 - 47.7465 = 358.0986$



	$R_X(au)$	$S_X(\omega)$
1	$cos\omega_c au$	$\pi[\delta(\omega-\omega_c)+\delta(\omega+\omega_c)]$
2	$\delta(au)$	1
3	$exp(-a \tau) \ a > 0$	$\frac{2a}{\omega^2+a^2}$
4	$\begin{cases} T(1 - \tau /T) & -T \le \tau \le T \\ 0 & otherwise \end{cases}$	$T^2 \left(\frac{\sin(\omega T/2)}{\omega T/2}\right)^2$
5	$(\omega_B/\pi)rac{\sin(\omega_B au)}{\omega_B au}$	$\begin{cases} 1 & -\omega_B \le \omega \le \omega_B \\ 0 & otherwise \end{cases}$

Weiner-Khinchin Relations 1.7

Weiner-Khinchin Relation states that the PSD and the autocorrelation function for wide-sense stationary random process are Fourier transform pair:

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j\omega\tau} d\tau$$
$$R_X(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) e^{j\omega\tau} d\omega$$

Let a rectangular function of time τ be

$$\left\{ \begin{array}{ll} T(1-|\tau|/T) & -T \leq \tau \leq T \\ 0 & otherwise \end{array} \right.$$

51. Use the property of an integrated unit-impulse function. (E.3; see Appendix E) to verify the property item 1 in table 4.1.

Solution:

$$S_X(\omega) = \pi[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)]$$

Show that

$$R_X(\tau) = cos\omega_c\tau$$

It is given that

$$\int_{-\infty}^{\infty} g(x)\delta(x-x_0)dx = g(x_0)$$

Based on the above relation

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) e^{j\omega t} d\omega &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \pi [\delta(\omega - \omega_c) + \delta(\omega + \omega_c)] e^{j\omega \tau} d\omega \\ &= \frac{1}{2} \left[e^{j\omega_c \tau} + e^{-j\omega_c \tau} \right] \\ &= \cos \omega_c \tau \end{aligned}$$

52. Use the property of an integrated unit-impulse function. (E.3; see Appendix E) to verify the property item 2 in table 4.1.

Solution:

 $S_X(\omega) = 1$



Show that

$$R_X(\tau) = \delta(\tau)$$

It is given that

$$\int_{-\infty}^{\infty} g(x)\delta(x-x_0)dx = g(x_0)$$

Based on the above relation

$$\int_{-\infty}^{\infty} R_X(\tau) e^{-j\omega t} d\tau = \int_{-\infty}^{\infty} \delta(\tau) e^{-j\omega \tau} d\omega = e^0$$
$$= 1$$

53. Use the property of an integrated unit-impulse function. (E.3; see Appendix E) to verify the property item 3 in table 4.1.

$$R_X(\tau) = exp(-a|\tau|) \ a > 0$$

Show that

$$S_X(\omega) = \frac{2a}{\omega^2 + a^2}$$

Solution: It is given that

$$\int_{-\infty}^{\infty} g(x)\delta(x-x_0)dx = g(x_0)$$

Based on the above relation

$$S_X(\omega) = \int_{-\infty}^0 e^{-a\tau} e^{-j\omega t} d\tau + \int_{-\infty}^\infty e^{a\tau} e^{-j\omega t} d\tau$$
$$= \int_{-\infty}^0 e^{(a-j\omega)\tau} d\tau + \int_0^\infty e^{-(a+j\omega)\tau} d\tau$$
$$= \left[\frac{e^{(a-j\omega)\tau}}{(a-j\omega)}\right]_{-\infty}^0 + \left[\frac{e^{(a+j\omega)\tau}}{(a+j\omega)}\right]_0^\infty$$
$$= \frac{1}{(a-j\omega)} - \frac{1}{(a+j\omega)} = \frac{2a}{(a^2+\omega^2)}$$



Multiplication of Two Random Process

Consider a two random process X(t) and Y(t) have the autocorrelation functions $R_X(\tau)$ and $R_Y(\tau)$, multiplication of two random process is

$$W(t) = X(t)Y(t)$$

The multiplication of autocorrelation functions is

$$R_W(\tau) = R_X(\tau)R_Y(\tau)$$

$$S_W(\omega) = \int_{-\infty}^{\infty} R_X(\tau) R_Y(\tau) d\tau$$

=
$$\int_{-\infty}^{\infty} R_X(\tau) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} S_Y(u) e^{ju\tau} du\right) e^{-j\omega\tau} d\tau$$

=
$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_Y(u) \int_{-\infty}^{\infty} R_X(\tau) e^{-j\omega\tau} d\tau$$

=
$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_Y(u) S_X(\omega - u) du = \frac{1}{2\pi} S_X(\omega) * S_Y(\omega)$$

56. The random process X(t) and Y(t) have the autocorrelation functions

$$R_X(\tau) = e^{(-10|\tau|)}$$
 and $R_Y(\tau) = 5\cos(600\tau)$

If Z(t) = X(t)Y(t), and if X(t) and Y(t) are independent , what is the PSD for Z(t) ? Solution:

$$R_X(\tau) = e^{(-10|\tau|)}$$

$$S_X(\omega) = \frac{20}{\omega^2 + 100}$$

$$R_Y(\tau) = 5\cos(600\tau)$$

$$S_Y(\omega) = 5\pi[\delta(\omega - 600) + \delta(\omega + 600)]$$

$$S_Z(\omega) = \frac{1}{2\pi} S_X(\omega) S_Y(\omega)$$

= $\frac{1}{2\pi} \int_{-\infty}^{\infty} 5\pi [\delta(\omega - u - 600) + \delta(\omega - u + 600)] \frac{20}{u^2 + 100} du$
= $50 \left(\frac{1}{(\omega - 600)^2 + 100} + \frac{1}{(\omega + 600)^2 + 100} \right)$

57. The random process X(t) and Y(t) have the autocorrelation functions

$$R_X(\tau) = e^{(-10|\tau|)}$$
 and $R_Y(\tau) = 6\cos(400\tau)$

If Z(t) = X(t)Y(t), and if X(t) and Y(t) are independent, what is the PSD for Z(t)? Solution:



$$R_X(\tau) = e^{(-10|\tau|)}$$

$$S_X(\omega) = \frac{20}{\omega^2 + 100}$$

$$R_Y(\tau) = 6\cos(400\tau)$$

$$S_Y(\omega) = 6\pi[\delta(\omega - 400) + \delta(\omega + 400)]$$

$$S_Z(\omega) = \frac{1}{2\pi} S_X(\omega) S_Y(\omega)$$

= $\frac{1}{2\pi} \int_{-\infty}^{\infty} 6\pi [\delta(\omega - u - 400) + \delta(\omega - u + 400)] \frac{20}{u^2 + 100} du$
= $60 \left(\frac{1}{(\omega - 400)^2 + 100} + \frac{1}{(\omega + 400)^2 + 100} \right)$

58. The random process X(t) and Y(t) have the autocorrelation functions

$$R_X(\tau) = e^{(-10|\tau|)}$$
 and $R_Y(\tau) = 7\cos(500\tau)$

If Z(t) = X(t)Y(t), and if X(t) and Y(t) are independent , what is the PSD for Z(t) ? Solution:

$$R_X(\tau) = e^{(-10|\tau|)}$$

$$S_X(\omega) = \frac{20}{\omega^2 + 100}$$

$$R_Y(\tau) = 7\cos(500\tau)$$

$$S_Y(\omega) = 7\pi[\delta(\omega - 500) + \delta(\omega + 500)]$$

$$S_Z(\omega) = \frac{1}{2\pi} S_X(\omega) S_Y(\omega)$$

= $\frac{1}{2\pi} \int_{-\infty}^{\infty} 7\pi [\delta(\omega - u - 500) + \delta(\omega - u + 500)] \frac{20}{u^2 + 100} du$
= $70 \left(\frac{1}{(\omega - 500)^2 + 100} + \frac{1}{(\omega + 500)^2 + 100} \right)$

59. A bandlimited wide-sense stationary random process X(t) has the PSD

$$S_X(\omega) = \begin{cases} 7\cos(\pi\omega/2\omega_M) & -\omega_M \le \omega \le \omega_M \\ 0 & otherwise \end{cases}$$

A carrier random process C(t) which is independent of X(t), is

$$C(t) = \sqrt{60}\cos(\omega_C t + \Theta)$$



where $\omega_C \gg \omega_M$, and where Θ is a random variale uniformly distributed between $\pm \pi$. If Y(t) = X(t)C(t), what is the PSD for Y(t)?

Solution:

$$C(t) = \sqrt{60}cos(\omega_C t + \Theta)$$

$$R_C(\tau) = 30cos(\omega_C \tau)$$

$$S_C(\omega) = 30\pi[\delta(\omega - \omega_C) + \delta(\omega + \omega_C)]$$

$$S_X(\omega) = 7cos(\pi u\omega/2\omega_M)$$

$$S_Y(\omega) = \frac{1}{2\pi} S_X(\omega) \star S_C(\omega)$$

= $\frac{1}{2\pi} \int_{-\infty}^{\infty} 7\cos(\pi u/2\omega_M) 30\pi [\delta(\omega - u - \omega_C) + \delta(\omega - u + \omega_C)] du$
= $105\cos(\pi(\omega - \omega_C)/2\omega_M) + 105\cos(\pi(\omega + \omega_C)/2\omega_M)$

60. A bandlimited wide-sense stationary random process X(t) has the PSD

$$S_X(\omega) = \begin{cases} 5\cos(\pi\omega/2\omega_M) & -\omega_M \le \omega \le \omega_M \\ 0 & otherwise \end{cases}$$

A carrier random process C(t) which is independent of X(t), is

$$C(t) = \sqrt{200}\cos(\omega_C t + \Theta)$$

where $\omega_C \gg \omega_M$, and where Θ is a random variale uniformly distributed between $\pm \pi$. If Y(t) = X(t)C(t), what is the PSD for Y(t)?

Solution:

$$C(t) = \sqrt{200\cos(\omega_C t + \Theta)}$$

$$R_C(\tau) = 100\cos(\omega_C \tau)$$

$$S_C(\omega) = 100\pi[\delta(\omega - \omega_C) + \delta(\omega + \omega_C)]$$

$$S_X(\omega) = 5\cos(\pi u \omega / 2\omega_M)$$

$$S_Y(\omega) = \frac{1}{2\pi} S_X(\omega) \star S_C(\omega)$$

= $\frac{1}{2\pi} \int_{-\infty}^{\infty} 5\cos(\pi u/2\omega_M) 100\pi [\delta(\omega - u - \omega_C) + \delta(\omega - u + \omega_C)] du$
= $250\cos(\pi(\omega - \omega_C)/2\omega_M) + 250\cos(\pi(\omega + \omega_C)/2\omega_M)$

61. A bandlimited wide-sense stationary random process X(t) has the PSD

$$S_X(\omega) = \begin{cases} 6\cos(\pi\omega/2\omega_M) & -\omega_M \le \omega \le \omega_M \\ 0 & otherwise \end{cases}$$

A carrier random process C(t) which is independent of X(t), is

$$C(t) = \sqrt{34}\cos(\omega_C t + \Theta)$$



where $\omega_C \gg \omega_M$, and where Θ is a random variale uniformly distributed between $\pm \pi$. If Y(t) = X(t)C(t), what is the PSD for Y(t)?

Solution:

$$C(t) = \sqrt{34}cos(\omega_C t + \Theta)$$

$$R_C(\tau) = 17cos(\omega_C \tau)$$

$$S_C(\omega) = 17\pi[\delta(\omega - \omega_C) + \delta(\omega + \omega_C)]$$

$$S_X(\omega) = 6cos(\pi u\omega/2\omega_M)$$

$$S_Y(\omega) = \frac{1}{2\pi} S_X(\omega) \star S_C(\omega)$$

= $\frac{1}{2\pi} \int_{-\infty}^{\infty} 6\cos(\pi u/2\omega_M) 17\pi [\delta(\omega - u - \omega_C) + \delta(\omega - u + \omega_C)] du$
= $51\cos(\pi(\omega - \omega_C)/2\omega_M) + 51\cos(\pi(\omega + \omega_C)/2\omega_M)$

62. The wide sense stationary random process X(t) has a PSD that is a constant $5 \times 10^{-6} V^2$ when |f| < 1kHz and is 0 otherwise. The random process $Y(t) = 10cos(\omega_0 t + \Theta)$ V where $f_O = 100 KHz$ and Θ is uniformly distributed between $\pm \pi$. X(t) and Y(t) are independent Z(t) = X(t)Y(t). What is the PSD for Z(t)?

Solution:

$$Y(t) = 10cos(\omega_o t + \Theta)$$

$$R_Y(\tau) = 50cos(\omega_o \tau)$$

$$S_Y(\omega) = 50\pi[\delta(\omega - \omega_o) + \delta(\omega + \omega_o)]$$

$$S_Z(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(u) \star S_Y(\omega)$$

= $\frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(u) S_Y(\omega - u) du$
= $\frac{1}{2\pi} \int_{-\infty}^{\infty} 5 \times 10^{-6} \times 50\pi [\delta(\omega - u - \omega_o) + \delta(\omega - u + \omega_o)] du$
= $125 \times 10^{-6} V^4 |\pm \omega_o - \omega| < 2\pi \times 10^3$

63. The wide sense stationary random process X(t) has a PSD that is a constant $7 \times 10^{-6} V^2$ when |f| < 0.9kHz and is 0 otherwise. The random process $Y(t) = 8cos(\omega_o t + \Theta)$ V where $f_O = 100 KHz$ and Θ is uniformly distributed between $\pm \pi$. X(t) and Y(t) are independent Z(t) = X(t)Y(t). What is the PSD for Z(t)?

Solution:

$$Y(t) = 8\cos(\omega_o t + \Theta)$$

$$R_Y(\tau) = 32\cos(\omega_o \tau)$$

$$S_Y(\omega) = 32\pi[\delta(\omega - \omega_o) + \delta(\omega + \omega_o)]$$



$$S_Z(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(u) \star S_Y(\omega)$$

= $\frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(u) S_Y(\omega - u) du$
= $\frac{1}{2\pi} \int_{-\infty}^{\infty} 7 \times 10^{-6} \times 32\pi [\delta(\omega - u - \omega_o) + \delta(\omega - u + \omega_o)] du$
= $112 \times 10^{-6} V^4 |\pm \omega_o - \omega| < 2\pi \times 10^3$

64. The wide sense stationary random process X(t) has a PSD that is a constant $6 \times 10^{-6} V^2$ when |f| < 1.1 kHz and is 0 otherwise. The random process $Y(t) = 9cos(\omega_o t + \Theta)$ V where $f_O = 100 KHz$ and Θ is uniformly distributed between $\pm \pi$. X(t) and Y(t) are independent Z(t) = X(t)Y(t). What is the PSD for Z(t) ?

Solution:

$$Y(t) = 9cos(\omega_o t + \Theta)$$

$$R_Y(\tau) = 40.5cos(\omega_o \tau)$$

$$S_Y(\omega) = 40.5\pi[\delta(\omega - \omega_o) + \delta(\omega + \omega_o)]$$

$$S_Z(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(u) \star S_Y(\omega)$$

= $\frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(u) S_Y(\omega - u) du$
= $\frac{1}{2\pi} \int_{-\infty}^{\infty} 6 \times 10^{-6} \times 40.5\pi [\delta(\omega - u - \omega_o) + \delta(\omega - u + \omega_o)] du$
= $121.5 \times 10^{-6} V^4$ $|\pm \omega_o - \omega| < 2\pi \times 10^3$



Linear Systems: 1.8

Consider a signal x(t) is passed through a filter with h(t), then the output of the filter y(t) is expressed as

$$y(t) = \int_0^\infty h(u)x(t-u)du$$

where h(t) is the impulse response of the system. Also above mentioned equation represents the convolution operation between input signal x(t) with h(t). The laplace transform the above system is

$$Y(s) = H(s)X(s)$$

where X(s) and Y(s) are the Laplace transforms of the input signal x(t) and output t(t) signals.

The above relation is applied for the random process X(t) and which expressed as

$$y(t) = \int_0^\infty h(u)X(t-u)du$$

1.8.1The mean of Y(t):

The mean of the output random process is

$$\mu_Y = E[Y(t)] = \int_0^\infty h(u)E[X(t-u)]du$$

$$\mu_Y = E[Y(t] = \mu_X \int_0^\infty h(u) du$$

The Laplace transform of h(t)

$$H(s) = \int_0^\infty h(t) e^{-st} dt$$

$$\mu_Y = E[Y(t] = \mu_X H(0)$$

1.8.2Cross-Correlating Y(t) and X(t):

Consider a two random process X(t) and Y(t) are wide sense stationary. When we consider both the random process it is called as jointly wide sense stationary. Then their cross-correlation function is defined as

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)] = \int_0^\infty h(u)E[X(t)X(t+\tau-u)]du$$
$$R_{XY}(\tau) = \int_0^\infty h(u)R_X(\tau-u)du$$
$$S_{XY}(j\omega) = \int_0^\infty h(u)\int_{-\infty}^\infty R_X(\tau-u)e^{-j\omega\tau}d\tau du$$

Let $v = \tau - u$

$$S_{XY}(j\omega) = \int_0^\infty h(u)e^{-j\omega u} \int_{-\infty}^\infty R_X(v)e^{-j\omega v}dv$$
$$= H(j\omega)S_X(\omega)$$



Autocorrelation of Y(t): 1.8.3

$$R_Y(\tau) = E[Y(t)Y(t+\tau)]$$

= $\int_0^\infty h(v) \int_0^\infty h(u)E[X(t-v)X(t+\tau-u)dudv]$
= $\int_0^\infty h(v) \int_0^\infty h(u)R_X(\tau+v-u)dudv$

$$S_Y(\omega) = \int_0^\infty h(v) \int_0^\infty h(u) \int_{-\infty}^\infty R_X(\tau + v - u) e^{-j\omega\tau} d\tau du dv$$

Let $w = \tau + v - u$

$$S_Y(\omega) = \int_0^\infty h(v)e^{j\omega v}dv \int_0^\infty h(u)e^{-j\omega u} \int_{-\infty}^\infty R_X(w)e^{-j\omega w}dw$$

= $H(-j\omega)H(j\omega)S_X(\omega)$
= $|H(j\omega)|^2S_X(\omega)$



65. Suppose in a given application

$$R_{XY}(\tau) = \begin{cases} aKe^{-a\tau} & \tau \ge 0\\\\ 0 & \tau \le 0 \end{cases}$$
$$S_{XY}(j\omega) = \frac{aK}{j\omega + a}$$

What are
$$R_{YX}(\tau)$$
 and $S_{YX}(j\omega)$ in this case? Interpret your results. [?]

Solution:

We Know that

$$R_{YX}(\tau) = R_{XY}(-\tau)$$

$$R_{YX}(\tau) = \begin{cases} aKe^{a\tau} & \tau \ge 0\\ \\ 0 & \tau \le 0 \end{cases}$$

And also

$$S_{YX}(j\omega) = S_{YX}^*(j\omega) = \frac{aK}{-j\omega + a}$$

66. Suppose that the PSD input to a linear system is $S_X(\omega) = K$. The cross-correlation of the input X(t) with the output Y(t) of the linear system is found to be

$$R_{XY}(\tau) = K \begin{cases} e^{-\tau} + 3e^{-2\tau} & \tau \ge 0\\ 0 & \tau \le 0 \end{cases}$$

What is the power filter function $|H(j\omega)|^2?$. [?]

Solution:

If $S_X(\omega) = K$ then

$$R_X(\tau) = K\delta(\tau)$$

$$R_{XY}(\tau) = \int_0^\infty h(u) R_X(\tau - u) du$$

$$R_{XY}(\tau) = K \int_0^{\infty} h(u)\delta(\tau - u)du$$

= $Kh(\tau)$

Similarly

$$h(t) = K \begin{cases} e^{-t} + 3e^{-2t} & t \ge 0 \\ 0 & t \le 0 \end{cases}$$

$$H(s) = \frac{1}{s+1} + \frac{3}{s+2} = \frac{(s+2) + 3(s+1)}{(s+1)(s+2)} = \frac{4s+5}{s^2+3s+2}$$



$$H(j\omega) = \frac{j4\omega+5}{-\omega^2+3j\omega+2}$$
$$|H(j\omega)|^2 = \frac{16\omega^2+25}{\omega^4+5\omega^2+4}$$

67. Suppose that the PSD input to a linear system is $S_X(\omega) = K$. The cross-correlation of the input X(t) with the output Y(t) of the linear system is found to be

$$R_{XY}(\tau) = K \begin{cases} 3e^{-\tau} + e^{-2\tau} & \tau \ge 0\\ 0 & \tau \le 0 \end{cases}$$

What is the power filter function $|H(j\omega)|^2$? . [?]

Solution:

If $S_X(\omega) = K$ then

$$R_X(\tau) = K\delta(\tau)$$

$$R_{XY}(\tau) = \int_0^\infty h(u) R_X(\tau - u) du$$

$$R_{XY}(\tau) = K \int_0^\infty h(u)\delta(\tau - u)du$$

= $Kh(\tau)$

Similarly

$$h(t) = K \begin{cases} 3e^{-t} + e^{-2t} & t \ge 0 \\ 0 & t \le 0 \end{cases}$$

$$H(s) = \frac{3}{s+1} + \frac{1}{s+2} = \frac{3(s+2) + (s+1)}{(s+1)(s+2)} = \frac{4s+7}{s^2 + 3s+2}$$

$$H(j\omega) \ = \ \frac{j4\omega+7}{-\omega^2+3j\omega+2}$$

$$|H(j\omega)|^2 = \frac{16\omega^2 + 49}{\omega^4 + 5\omega^2 + 4}$$

68. Suppose that the PSD input to a linear system is $S_X(\omega) = K$. The cross-correlation of the input X(t) with the output Y(t) of the linear system is found to be

$$R_{XY}(\tau) = K \begin{cases} 2e^{-2\tau} + 3e^{-\tau} & \tau \ge 0\\ 0 & \tau \le 0 \end{cases}$$
nction $|H(i\omega)|^2$? [?]

What is the power filter function $|H(j\omega)|^{\rm 2}?$. [•]



Solution:

If $S_X(\omega) = K$ then

$$R_X(\tau) = K\delta(\tau)$$

$$R_{XY}(\tau) = \int_0^\infty h(u)R_X(\tau - u)du$$

$$R_{XY}(\tau) = K\int_0^\infty h(u)\delta(\tau - u)du$$

$$= Kh(\tau)$$

Similarly

$$h(t) = K \begin{cases} 2e^{-t} + 3e^{-t} & t \ge 0 \\ 0 & t \le 0 \end{cases}$$

$$H(s) = \frac{2}{s+2} + \frac{3}{s+1} = \frac{2(s+1) + 3(s+2)}{(s+1)(s+2)} = \frac{5s+8}{s^2+3s+2}$$
$$H(j\omega) = \frac{j5\omega+8}{-\omega^2+3j\omega+2}$$

$$|H(j\omega)|^2 = \frac{25\omega^2 + 64}{\omega^4 + 5\omega^2 + 4}$$

69. The PSD of the random process X(t) and the transfer function of a network are

$$S_X(\omega) = \frac{1}{\omega^2 + (100)^2} \text{ and } H(s) = \frac{s}{(s+10)(s+9000)}$$
$$Y(s) = H(s)X(s). \text{ Find } \mu_Y, S_{XY}(j\omega) \text{ and } S_Y(\omega) \text{ . [?]}$$

Solution:

 $S_X(\omega)$ doesn't have any dc component (unit impulse function) when $\omega = 0$, hence

$$\mu_X = 0$$

Also the relation is

$$\mu_Y = \mu_X H(0) = 0$$

$$S_{XY}(j\omega) = H(j\omega)S_X(\omega)$$

= $\frac{j\omega}{(\omega^2 + (100)^2)(j\omega + 10)(j\omega + 9000)}$

$$S_Y(\omega) = |H(j\omega)|^2 S_X(\omega)$$

= $\frac{\omega^2}{(\omega^2 + (100)^2)(\omega^2 + (10)^2)(\omega^2 + (9000)^2)}$



70. The PSD of the random process X(t) and the transfer function of a network are

$$S_X(\omega) = \frac{1}{\omega^2 + (80)^2}$$
 and $H(s) = \frac{s}{(s+9)(s+10000)}$

Y(s) = H(s)X(s). Find μ_Y , $S_{XY}(j\omega)$ and $S_Y(\omega)$. [?]

Solution:

 $S_X(\omega)$ doesn't have any dc component (unit impulse function) when $\omega = 0$, hence

 $\mu_X = 0$

Also the relation is

$$\mu_Y = \mu_X H(0) = 0$$

$$S_{XY}(j\omega) = H(j\omega)S_X(\omega)$$

=
$$\frac{j\omega}{(\omega^2 + (80)^2)(j\omega + 9)(j\omega + 10000)}$$

$$S_Y(\omega) = |H(j\omega)|^2 S_X(\omega)$$

= $\frac{\omega^2}{(\omega^2 + (80)^2)(\omega^2 + (9)^2)(\omega^2 + (10000)^2)}$

71. The PSD of the random process X(t) and the transfer function of a network are

$$S_X(\omega) = \frac{1}{\omega^2 + (70)^2}$$
 and $H(s) = \frac{s}{(s+10)(s+11000)}$

Y(s) = H(s)X(s). Find μ_Y , $S_{XY}(j\omega)$ and $S_Y(\omega)$. [?]

Solution:

 $S_X(\omega)$ doesn't have any dc component (unit impulse function) when $\omega = 0$, hence

$$\mu_X = 0$$

Also the relation is

$$\mu_Y = \mu_X H(0) = 0$$

$$S_{XY}(j\omega) = H(j\omega)S_X(\omega)$$

= $\frac{j\omega}{(\omega^2 + (70)^2)(j\omega + 10)(j\omega + 11000)}$

$$S_Y(\omega) = |H(j\omega)|^2 S_X(\omega)$$

= $\frac{\omega^2}{(\omega^2 + (70)^2)(\omega^2 + (10)^2)(\omega^2 + (11000)^2)}$

