Point-to-Point Communication[1]

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Point-to-point communication: [1]

- Slides are prepared to use in class room purpose, may be used as a reference material
- All the slides are prepared based on the reference material
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- This material is not for commercial purpose.
- This material is prepared based on Wireless Communication for M Tech in DECS course as per Visvesvaraya Technological University (VTU) syllabus (Karnataka State, India).

Point-to-point communication

- Detection in a Rayleigh fading channel
  1. Non-coherent detection
  2. Coherent detection
  3. From BPSK to QPSK: exploiting the degrees of freedom
- Diversity
  1. Time diversity
  2. Repetition coding
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Notation

$\Re$	Real numbers
$\Im$	Imaginary numbers
$\mathcal{C}$	Complex numbers
$h[m]$	Scalar channel, complex valued, at time m
h^*	Complex conjugate of the complex valued scalar h
$\mathcal{N}(\mu,\ \sigma^2)$	Real Gaussian random variable with mean μ and variance σ^2
$\mathcal{CN}(0,\ \sigma^2)$	Circularly symmetric complex Gaussian random variable: the real and imaginary parts are i.i.d $\mathcal{N}(0,\ \sigma^2)$
i.i.d.	independent and identically distributed

Gaussian random variable x is of the form

$$x = \sigma w + \mu$$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

The mean of x is μ and the variance is equal to σ^2. This is denoted as $\mathcal{N}(\mu,\ \sigma^2)$
The standard Gaussian random variable is denoted by $\mathcal{N}(0,\ 1)$.

$$f(x) = \frac{1}{\sqrt{2\pi}} exp\left(-\frac{x^2}{2}\right)$$

$x_i \sim \mathcal{N}(\mu,\ \sigma^2)$ where the $\sim$ notation represents the phrase "is distributed as"

Non coherent detection

- Consider a flat fading model where the channel can be represented by a single discrete-time complex filter $h_0[m]$, which is abbreviated as h[m]

$$y[m] = x[m]\,h[m] + w[m] \quad (1)$$

- where $w[m] \sim \mathcal{CN}(0,\ N_0)$ and channel is Rayleigh fading, i.e.$h[m] \sim \mathcal{CN}(0,\ 1)$.
- Consider uncoded binary antipodal signaling (binary phase-shift keying, BPSK) with amplitude a, i.e., $x[m] = \pm a$, and the symbols $x[m]$ are independent over time.
- This signaling scheme fails completely, even in the absence of noise, since the phase of the received signal $y[m]$is uniformly distributed between 0 and 2π regardless of whether $x[m] = a$ or $x[m] = -a$ is transmitted.

Consider an orthogonal modulation scheme: a form of binary pulse-position modulation.

$$X_A = \begin{pmatrix} x[0] \\ x[1] \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} \quad (2)$$

or

$$X_B = \begin{pmatrix} x[0] \\ x[1] \end{pmatrix} = \begin{pmatrix} 0 \\ a \end{pmatrix} \quad (3)$$

The detection is based on

$$Y = \begin{pmatrix} y[0] \\ y[1] \end{pmatrix} \quad (4)$$

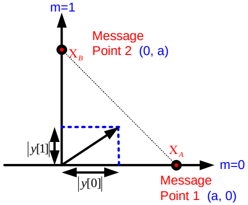


Figure 1: Signal space diagram.

Detection in a Rayleigh fading channel

This is a simple hypothesis testing problem, and it is straightforward to derive the maximum likelihood (ML) rule

$$\Delta(y) \frac{\geq^{x_a}}{<_{x_b}} 0 \tag{5}$$

where y is the log-likelihood ratio

$$\Delta(y) = \ln\left\{\frac{f(y/x_a)}{f(y/x_b)}\right\} \tag{6}$$

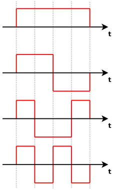


Figure 2: Orthogonal signals.

- If X_A is transmitted, then $y[0]\ \mathcal{CN}(0, a^2 + N_0)$, and $y[1]\ \mathcal{CN}(0, N_0)$ and y[0],y[1] are independent
- Similarly, if X_B is transmitted,$y[0]\ \mathcal{CN}(0, N_0)$ and $y[1]\ \mathcal{CN}(0, a^2 + N_0)$.
- log-likelihood ratio can be computed to be

$$\Delta(y) = \frac{\left\{|y\,[0]|^2 - |y\,[1]|^2\right\} a^2}{(a^2 + N_0)\, N_0} \tag{7}$$

- The detector is called as an energy or square-law detector. The optimal detector does not depend on how $h[0]$and $h[1]$ are correlated.

- The probability of error:

$$p_e = P\left\{|y\,[1]|^2 > |y\,[0]|^2\, x_a\right\} = \left[2 + \frac{a^2}{N_0}\right]^{-1} = \left[2\left(1 + \frac{a^2}{2N_0}\right)\right]^{-1} \tag{8}$$

- SNR is defined as

$$SNR = \frac{average\ received\ signal\ energy\ per\ (complex)\ symbol\ time}{noise\ energy\ per\ (complex)\ symbol\ time} \tag{9}$$

$$SNR = \frac{a^2}{2N_0} \tag{10}$$

- The probability of error is given by

$$p_e = \frac{1}{2(1 + SNR)} \tag{11}$$

Coherent detection over AWGN channel without fading

- It is instructive to compare its performance with detection in an AWGN channel without fading:

$$y\left[m\right] = x\left[m\right] + w\left[m\right] \tag{12}$$

- Detection in a Rayleigh fading channel, For antipodal signaling (BPSK),$x[m] \pm a$ sufficient statistic is $\Re y[m]$ and the error probability is

$$p_e = Q\left(\frac{a}{\sqrt{N_0/_2}}\right) = Q\left(\sqrt{2SNR}\right) \tag{13}$$

- This function decays exponentially with x^2 more specifically.

$$Q\left(x\right) < e^{-x^2/2}, x > 0 \tag{14}$$

$$Q\left(x\right) > \frac{1}{\sqrt{2\pi}x}\left(1 - \frac{1}{x^2}\right)e^{-x^2/2}, x > 1 \tag{15}$$

- Thus,the detection error probability decays exponentially in SNR in the AWGN channel while it decays only inversely with the SNR in the fading channel.
- To get an error probability of 10^{-3}, an SNR of only about 7 dB is needed in an AWGN channel (as compared to 27 dB in the non-coherent fading channel).
- Note that $2\sqrt{SNR}$ is the separation between the two constellation points as a multiple of the standard deviation of the Gaussian noise.

The expectation (mean) of the random variable X_1

$$E[X_1] = E[-a + w_1] = -a + 0 = -a \quad \Longrightarrow \quad \mu$$

The variance of the random variable X_1

$$Var[X_1] = Var[-a + w_1] = 0 + \frac{N_0}{2} = \frac{N_0}{2} \quad \Longrightarrow \quad \sigma^2$$

Conditional pdf of random variable X_1 given that symbol 0 is transmitted is given by

$$f_{X_1}(x_1|0) = \frac{1}{\sqrt{\pi N_0}} \exp\left[\frac{-(x_1+a)^2}{N_0}\right] \qquad f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

When symbol 0 is transmitted, an error will occur, if $x_1 > 0$ in which case a decision is made in favor of symbol 1 $P_e(0) = P(x_1 > 0|\text{symbol 0 is transmitted})$ $P_e(0)$ can be computed by integrating conditional pdf $f_{X_1}(x_1|0)$

$$P_e(0) = \int_0^\infty f_{X_1}(x_1|0)dx_1 = \frac{1}{\sqrt{\pi N_0}} \int_0^\infty \exp\left[\frac{-(x_1+a)^2}{N_0}\right] dx_1$$

$$P_e(0) = \frac{1}{\sqrt{\pi N_0}} \int_0^\infty \exp\left[\frac{-(x_1+a)^2}{N_0}\right] dx_1$$

$$Let \quad u = \frac{(x_1+a)}{\sqrt{N_0/2}}$$

$$dx_1 = \sqrt{N_0/2}du$$

When the lower is 0 then it is $\sqrt{\frac{2}{N_0}}a$ when upper limit is ∞ it is ∞

$$P_e(0) = \frac{1}{\sqrt{\pi N_0}} \int_{\sqrt{\frac{2}{N_0}}a}^{\infty} \exp\left[-u^2/2\right] \sqrt{N_0/2}du = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\frac{2}{N_0}}a}^{\infty} \exp\left[-u^2/2\right] du$$

$$Q(u) = \frac{1}{\sqrt{2\pi}} \int_{u}^{\infty} \exp(-x^2/2)dx$$

By comparing with Q function

$$P_e(0) = \sqrt{\frac{2}{N_0}a}$$

Similarly when symbol 1 is transmitted, if error occurs, then a decision is made in favor of symbol 0. The probability of error when symbol 1 is transmitted is $P_e(1) = P(x_1 < 0|\text{symbol } 1$ is transmitted. $P_e(0)$ can be computed by integrating conditional pdf $f_{X_1}(x_1|0)$

$$P_e(1) = \sqrt{\frac{2}{N_0}a}$$

Symbols 0 and 1 are equiprobable, that is $P_e(0) = P_e(1) = 1/2$ then the average probability of symbol error is

$$P_e = \frac{1}{2}[P_e(0) + P_e(1)] = \sqrt{\frac{2}{N_0}a}$$

$$P_e = \sqrt{\frac{2}{N_0}a}$$

- Knowing the channel gains,coherent detection of BPSK can now be performed on a symbol by symbol basis. We can focus on one symbol time and drop the time index

$$y = hx + w \tag{16}$$

- Detection of x from y can be done in a way similar to that in the AWGN case; the decision is now based on the sign of the real sufficient statistic

$$r = \Re\{(h/|h|)^* y\} = |h|x + Z \tag{17}$$

- where $z \sim N(0, N_0/2)$.
- If the transmitted symbol is $x = \pm a$, then, for a given value of h, the error probability of detecting x is

$$Q\left(\frac{a|h|}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{2|h|^2 SNR}\right) \tag{18}$$

- where $SNR = a^2/N_0$ is the average received signal-to-noise ratio per symbol time.
- For Rayleigh fading when h=eN(0,1) direct integration yields

$$p_e = E\left[Q\left(\sqrt{2\,|h|^2\,SNR}\right)\right] = \frac{1}{2}\left(1 - \sqrt{\frac{SNR}{1+SNR}}\right) \tag{19}$$

- At high SNR, Taylor series expansion yields

$$(1+x)^{0.5} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}$$

$$(1+x)^{-0.5} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}$$

$$\sqrt{\frac{SNR}{1+SNR}} = 1 - \frac{1}{2SNR} + o\left(\frac{1}{SNR^2}\right)$$

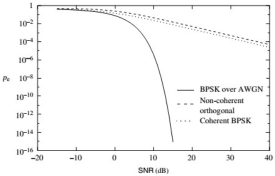


Figure 3: Orthogonal signals.

- Substituting into previous equation

$$p_e \approx \frac{1}{4SNR}$$

Conclusions: The following conclusions were made

- Probability of error which decays inversely proportional to the SNR, similar to the non-coherent orthogonal signaling scheme.
- There is only a 3 dB difference in the required SNR between the coherent and non-coherent schemes in contrast, at an error probability of 10^{-3}, there is a 17 dB difference between the performance on the AWGN channel and coherent detection on the Rayleigh fading channel

- The detection in the fading channel has a poor performance because the channel gain is random and it is in a "deep fade"
- The quantity $|h|^2 SNR$ is the instantaneous received SNR.
- Under typical channel conditions, i.e.,$|h|^2 SNR \gg 1$, the conditional error probability is very small, since the tail of the Q-function decays very rapidly.
- In this regime, the separation between the constellation points is much larger than the standard deviation of the Gaussian noise
- On the other hand, when h square SNR is of the order of 1 or less, the separation is of the same order as the standard deviation of the noise and the error probability becomes significant. The probability of this event is

$$p\left\{|h|^2\ SNR < 1\right\} = \int_0^{1/SNR} e^{-x} dx \\ = \frac{1}{SNR} + o\left(\frac{1}{SNR^2}\right) \tag{20}$$

Conclusions: The following conclusion is made

- At high-SNR error events most often occur because the channel is in deep fade and not as a result of the additive noise being large.
- In contrast, in the AWGN channel the only possible error mechanism is for the additive noise to be large.
- Thus, the error probability performance over the AWGN channel is much better.

QPSK(Quadrature Phase-Shift-Keying)

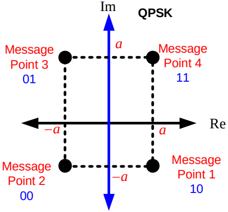
- BPSK modulation, $x[m] = \pm a$ uses only the real dimension (the I channel).
- To increase the spectral efficiency both the I and Q channels are used simultaneously in coherent communication using QPSK modulation, and its constellation is

$$\{a(1+j), a(1-j), a(-1+j), a(-1-j)\}; \quad (21)$$

- A BPSK symbol is transmitted on each of the I and Q channels simultaneously.

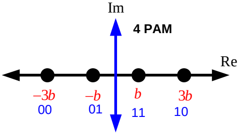
The noise is independent across the I and Q channels, the bits can be detected separately and the bit error probability in the AWGN channel is

$$Q\left(\sqrt{\frac{2a^2}{N_0}}\right) \quad (22)$$



For BPSK, the SNR is given by

$$SNR = \frac{a^2}{N_0} \quad (23)$$



while for QPSK

$$SNR = \frac{2a^2}{N_0} \quad (24)$$

- The error probability of QPSK under Rayleigh fading can be similarly obtained by replacing SNR by SNR/2 in the corresponding expression 18 for BPSK to yield

$$p_e = \frac{1}{2}\left(1 - \sqrt{\frac{SNR}{2+SNR}}\right) \approx \frac{1}{2SNR} \tag{25}$$

- The constellation 4 PAM SYMBOL is {-3b,-b,b,3b} and the average error probability on the AWGN channel is

$$\frac{3}{2}Q\left(\sqrt{\frac{2b^2}{N_0}}\right) \tag{26}$$

- To achieve approximately the same error probability as QPSK, the argument inside the Q-function should be the same as that in 23 and hence b should be the same as a, i.e., the same minimum separation between points in the two constellations.
- But QPSK requires a transmit energy of $2a^2$ per symbol, while 4-PAM requires a transmit energy of $5b^2$ per symbol.
- Hence, for the same error probability, approximately 2.5 times more transmit energy is needed: a 4 dB worse performance
- The loss is due to the fact that it is more energy efficient to pack, for a desired minimum distance separation, a given number of constellation points in a higher-dimensional space than in a lower-dimensional space.

- the BPSK information symbol is $u[m]$at time $m(u[m] = \pm 1)$, the transmitted symbol at time m is given by

$$x[m] = u[m]\, x[m-1] \tag{27}$$

- But since non-coherent orthogonal modulation also has a 3-dB worse performance compared to coherent BPSK, this implies that differential BPSK and non-coherent orthogonal modulation have the same error probability performance

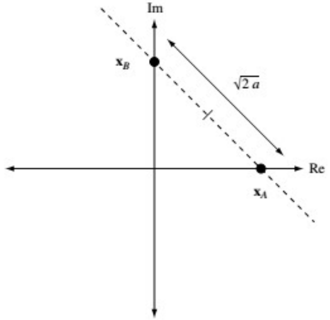


Figure 4: Geometry of orthogonal modulation

- On the other hand, differential BPSK conveys one bit of information and uses one real dimension per single symbol time, and therefore has twice the spectral efficiency of orthogonal modulation.

Diversity

- The performance of the various modulation schemes are all bad for fading channels and their error probabilities decay very slowly, like 1/SNR.
- The root cause of this poor performance is that reliable communication depends on the strength of a single signal path.
- There is a significant probability that this path will be in a deep fade and suffer from errors
- A natural solution to improve the performance is to ensure that the information symbols pass through multiple signal paths, each of which fades independently, making sure that reliable communication is possible as long as one of the paths is strong.
- Diversity technique can **improves** the performance over fading channels.
- The following are the diversity technique used in deep fading channels.
  - **Time Diversity**
  - **Frequency Diversity**
  - **Space Diversity**

Time Diversity

- Time diversity is achieved by averaging the fading of the channel over time.
- Typically, the channel coherence time is of the order of tens to hundreds of symbols, and therefore the channel is highly correlated across consecutive symbols.
- The coded symbols are transmitted through independent fading gains using, interleaving of codewords as shown in Figure 5.

- Consider a codeword $\mathbf{X} = [x_1, x_2, \dots x_L]^T$ of length L symbols are transmitted and the received signal is:

$$y_\ell = h_\ell x_\ell + w_\ell, \ell = 1......L$$

where L is the number of diversity branches.

- Assuming ideal interleaving so that consecutive symbols x_ℓ are transmitted sufficiently far apart in time, and assuming that the h_ℓ are independent.

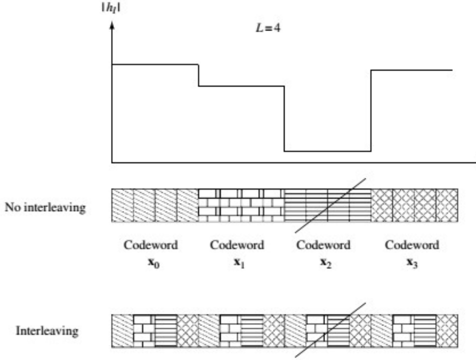


Figure 5: Interleaving

- Consider a code in which each code-word contains four code symbols[2].
- Suppose there are 16 symbols, which correspond to four code-words.
- That is, code symbols from 1 to 4 form a code-word, from 5 to 8 another codeword, and so on.
- In block interleaving, first creates a 4X4 2-D array, called block interleaver as shown in Figure6.
- The 16 code symbols are read into the 2-D array in a column-by-column (or row-by-row) manner.
- The interleaved code symbols are obtained by writing the code symbols out of the 2-D array in a row-by-row (or column by-column) fashion.
- This process has been depicted in Figure6.
- Assume a burst of errors involving four consecutive symbols as shown in Figure Figure6 with shades.
- After de-interleaving as shown in Figure6, the error burst is effectively spread among four code-words, resulting in only one code symbol in error for each of the four code-words

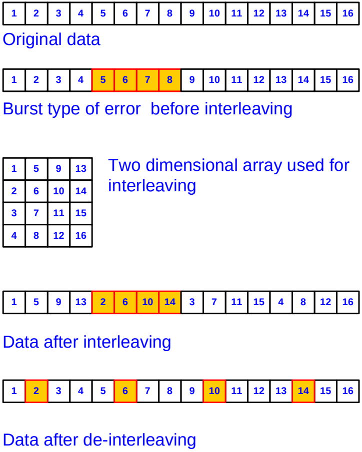


Figure 6: Block Interleaving

[2]

Repetition coding

- The simplest code is a repetition code, in which $x_\ell = x_1$ for $\ell = 1....L$
- In vector form, the overall received signal becomes

$$\mathbf{y} = \mathbf{h}x_1 + \mathbf{w}$$

where $\mathbf{y} = [y_1, y_2.....y_L]^T$ $\mathbf{h} = [h_1, h_2.....h_L]^T$ $\mathbf{y} = [w_1, w_2.....w_L]^T$

- Consider a **coherent detection** of x_1, i.e., the **channel gains are known to the receiver**.
- This is the canonical vector Gaussian detection problem, the scalar form is

$$\frac{\mathbf{h}^*}{||\mathbf{h}||}\mathbf{y} = ||\mathbf{h}||\mathbf{x_1} + \frac{\mathbf{h}^*}{||\mathbf{h}||}\mathbf{w} \tag{28}$$

- The receiver structure is a matched filter and is also called a maximal ratio combiner: it weighs the received signal in each branch in proportion to the signal strength and also aligns the phases of the signals in the summation to maximize the output SNR. This receiver structure is also combining.
- The error probability, conditional on $\mathbf{h}$ for BPSK modulation with $x_1 = \pm a$ is,

$$Q\left(\sqrt{2||\mathbf{h}||^2\mathbf{SNR}}\right) \tag{29}$$

- where $SNR = a^2/N0$ is the average received signal-to-noise ratio per (complex) symbol time.

- Under Rayleigh fading with each gain h_ℓ **i.i.d** $\mathcal{CN}(0,1)$

$$||\mathbf{h}||^2 = \sum_{l=1}^{L} |h_\ell|^2 \tag{30}$$

- is a sum of the squares of 2L independent real Gaussian random variables, each term $h\ell$ square being the sum of the squares of the real and imaginary parts of $h\ell$.
- It is Chi-square distributed with 2L degrees of freedom, and the density is given by

$$f(x) = \frac{1}{(L-1)!} x^{L-1} e^{-x}, \quad x \geq 0 \tag{31}$$

- The average error probability can be explicitly computed to be

$$p_{e=} \int_0^a Q\sqrt{2 * SNR}\; f(x)dx \tag{32}$$

$$= \left(\frac{1-\mu}{2}\right)^L \sum_{l=0}^{L-1} \begin{pmatrix} L-1+l \\ l \end{pmatrix} \left(\frac{1+\mu}{2}\right)^l \tag{33}$$

$$\mu = \sqrt{\frac{SNR}{1+SNR}} \tag{34}$$

- The error probability as a function of the SNR for different numbers of diversity branches L is plotted in Figure 7.
- Increasing L dramatically decreases the error probability.
- At high SNR, we can see the role of L analytically: consider the leading term in the Taylor series expansion in 1/SNR to arrive at the approximations
- The error probability decreases as the Lth power of SNR, corresponding to a slope of -L in the error probability curve.

$$\frac{1+\mu}{2} \approx 1, \quad \textit{and} \qquad \frac{1-\mu}{2} = \frac{1}{4SNR}$$

$$\sum_{l=0}^{L-1} \begin{pmatrix} L-1+l \\ l \end{pmatrix} = \begin{pmatrix} 2L-1 \\ L \end{pmatrix}$$

$$p_e \approx \begin{pmatrix} 2L-1 \\ L \end{pmatrix} \frac{1}{(4SNR)^L}$$

$$\begin{pmatrix} n \\ r \end{pmatrix} = \frac{n!}{r!(n-r)!}$$

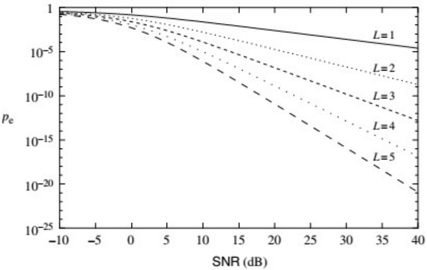


Figure 7: Plot of P_e V/S SNR

- The typical error event at high SNR is when the overall channel gain is small.
- This happens with probability

$$p\left\{\|h\|^2 < {}^1\!/_{SNR}\right\}$$

- For small x, the probability density function

$$f(x) \approx \frac{1}{(L-1)!}x^{L-1}$$

$$p\left\{||h||^2 < \frac{1}{SNR}\right\} = \approx \int_0^{\frac{1}{SNR}} \frac{1}{(L-1)!}x^{L-1}dx = \frac{1}{L!}\frac{1}{SNR^L}$$

Typically L is called the diversity gain of the system.

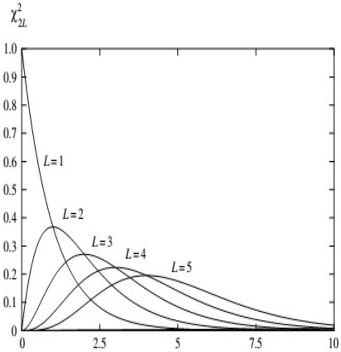


Figure 8: The pdf of h

Beyond repetition coding

- The repetition code repeats the same symbol over the L symbol times and achieves a diversity gain but it **does not exploit the degrees of freedom** available in the channel effectively.
- By using more sophisticated codes, a **coding gain** and **diversity gain** obtained.
- Consider a **rotation code** to explain some of the issues in code design for fading channels.
- Consider the case L=2. A repetition code which repeats a BPSK symbol $u = \pm a$ twice obtains a diversity gain of 2 but would only transmit one bit of information over the two symbol times.
- Transmitting two independent BPSK symbols $u_1 u_2$ over the two times would use the available degrees of freedom more efficiently, but of course offers **no diversity gain**: an error would be made whenever one of the two channel gains h1 h2is in deep fade.
- To get **both benefits**, consider instead a scheme that transmits the vector

$$\mathbf{X} = \mathbf{R}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad where \quad \mathbf{R} := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\mathbf{X_A} = \mathbf{R}\begin{bmatrix} a \\ a \end{bmatrix} \quad \mathbf{X_B} = \mathbf{R}\begin{bmatrix} -a \\ a \end{bmatrix} \quad \mathbf{X_C} = \mathbf{R}\begin{bmatrix} -a \\ -a \end{bmatrix} \quad \mathbf{X_D} = \mathbf{R}\begin{bmatrix} a \\ -a \end{bmatrix}$$

- The received signal is given by

$$y_\ell = h_\ell x_\ell + w_\ell, \qquad \ell = 1, 2, 3 \tag{35}$$

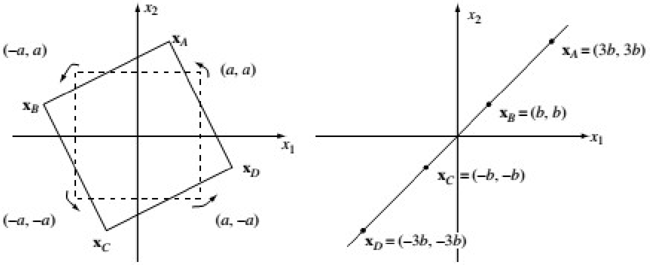


Figure 9: Rotation and repetition codes

- The error probability is derived in terms of **union bound**.
- Consider a code $\mathbf{X}_A$ is transmitted then the union bound is

$$p_e \leq P\left\{X_A \rightarrow X_B\right\} + P\left\{X_A \rightarrow X_C\right\} + P\left\{X_A \rightarrow X_B\right\}$$

- conditioned on the channel gains h_1 and h_2,

$$\mathbf{u_A} = \mathbf{R}\begin{bmatrix} h_1 x_{A1} \\ h_2 x_{A2} \end{bmatrix} \quad \mathbf{u_B} = \mathbf{R}\begin{bmatrix} h_1 x_{B1} \\ h_2 x_{B2} \end{bmatrix}$$

$$\begin{aligned} P\left\{X_A \rightarrow X_B | h_1, h_2|\right\} &= Q\left(\frac{||U_A - U_B||}{2\sqrt{N_0/2}}\right) \\ &= Q\left(\sqrt{\frac{SNR(|h_1|^2|d_1|^2 + |h_2|^2|d_2|^2)}{2}}\right) \end{aligned}$$

where $SNR = a^2/N_0$ and

$$d = \frac{1}{a}(X_A - X_B) = \begin{bmatrix} 2\cos\theta \\ 2\sin\theta \end{bmatrix} \tag{36}$$

- is the normalized difference between the codewords, normalized such that the transmit energy is 1 per symbol time. We use the upper bound

$$Q(x) \leq e^{-x^2/2} \quad for \quad x > 0 \tag{37}$$

$$P\left\{X_A \rightarrow X_B | h_1, h_2|\right\} \leq exp\left(\frac{-SNR(|h_1|^2|d_1|^2 + |h_2|^2|d_2|^2)}{4}\right)$$

Averaging with respect to h1 and h2 under the independent Rayleigh fading assumption, we get

$$P\left\{X_A \to X_B\right\} \leq E_{h_1,h_2}\left[\exp\left(\frac{-SNR\left(|h_1|^2|d_1|^2+\right)|h_2|^2|d_2|^2}{4}\right)\right] \\ = \left(\frac{1}{1+SNR|d_1|^2/4}\right)\left(\frac{1}{1+SNR|d_2|^2/4}\right) \tag{38}$$

- Here we have used the fact that the moment generating function for a unit mean exponential random variable X is

$$E\left[e^{sX}\right] = 1/(1-S) \tag{39}$$

- We first observe that if d1=0 or d2=0, then the diversity gain of the code is only 1. If they are both non-zero, then at high SNR the above bound on the pairwise error probability becomes

$$P\{X_A \rightarrow X_B\} \leq \frac{16}{|d_1 d_2|^2} SNR^{-2} \tag{40}$$

call

$$\delta_{AB} = |d_1 d_2|^2 \tag{41}$$

the squared product distance between xA and xB , when the average energy of the code is normalized to be 1 per symbol time.

This determines the pairwise error probability between the two codewords.

- Similarly, we can define delta ij to be the squared product distance between xi and xj , i j = A B C D. Combining (3.55) with (3.49) yields a bound on the overall error probability:

$$P_e \leq 16\left(\frac{1}{\delta_{AB}}+\frac{1}{\delta_{AC}}+\frac{1}{\delta_{AD}}\right) SNR^{-2} \tag{42}$$

$$\leq \frac{48}{\min_{j=B,C,D}\delta_{Aj}} SNR^{-2} \tag{43}$$

We see that as long as ij ¿0 for all i j, we get a diversity gain of 2.

- This parameter depends on theta, and we can optimize over to maximize the coding gain. Here

$$\delta_{AB} = \delta_{AD} = 4\sin^2 2\theta \quad (44)$$

and

$$\delta_{AC} = 16\cos^2 2\theta \quad (45)$$

- The angle theta* that maximizes the minimum squared product distance makes delta AB equal delta AC, yielding

$$\theta^* = \left(\frac{1}{2}\right)\tan^{-1} 2 and \min \delta_{ij} = \frac{16}{5} \quad (46)$$

- The bound in now becomes

$$P_e \leq 15SNR^{-2} \quad (47)$$

- To get more insight into why the product distance is important, we see from 46 that the typical way for xA to be confused with xB is for the squared Euclidean distance

$$\left(|h_1|^2 |d_1|^2 + |h_2|^2 |d_2|^2\right) \tag{48}$$

- between the received codewords to
  be of the order of 1/SNR. This event holds roughly when both

$$|h_1|^2 |d_1|^2 \tag{49}$$

and

$$|h_2|^2 |d_2|^2 \tag{50}$$

are of the order of 1/SNR, and this happens with probability approximately

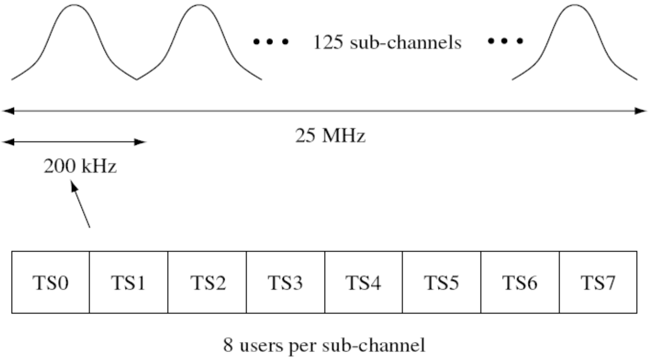
$$\left(\frac{1}{|d_1|^2 SNR}\right)\left(\frac{1}{|d_2|^2 SNR}\right) = \frac{1}{|d_1|^2 |d_2|^2} SNR^{-2} \tag{51}$$

- It is interesting to see how this code compares to the repetition scheme. To keep the bit rate the same (2 bits over 2 real-valued symbols), the repetition scheme would be using 4-PAM modulation -3b -b b 3b.
- the pairwise error probability between two adjacent codewords (say,xA and xB)is

$$P\left\{X_A \rightarrow X_B\right\} = E\left[Q\left(\sqrt{\frac{SNR\left(\left|h_1\right|^2\left|d_1\right|^2+\left|h_2\right|^2\left|d_2\right|^2\right)}{2}}\right)\right] \tag{52}$$

Time diversity in GSM

- Global System for Mobile (GSM) is a digital cellular standard developed in Europe in the 1980s.
- GSM is a frequency division duplex (FDD) system and uses two 25-MHz bands, one for the uplink (mobiles to base-station) and one for the downlink (base-station to mobiles).
- The original bands set aside for GSM are the 890-915MHz band (uplink) and the 935-960MHz band (downlink).
- The bands are further divided into 200-kHz sub-channels and each sub-channel is shared by eight users in a time-division fashion (time-division multiple access (TDMA)).
- The data of each user are sent over time slots of length 577 microseconds (μs) and the time slots of the eight users together form a frame of length 4.615 ms (Figure 3.9).



- Voice is the main application for GSM. Voice is coded by a speech encoder into speech frames each of length 20 ms.
- The bits in each speech frame are encoded by a convolutional code of rate 1/2, with the two generator polynomials D4 +D3+1 and D4 +D3+D+1.
- The number of coded bits for each speech frame is 456.
- To achieve time diversity, these coded bits are interleaved across eight consecutive time slots assigned to that specific user: the 0th, 8th, . . . , 448th bits are put into the first time slot, the 1st, 9th, . . . , 449th bits are put into the second time slot, etc.
- Since one time slot occurs every 4.615 ms for each user, this translates into a delay of roughly 40 ms, a delay judged tolerable for voice.
- The eight time slots are shared between two 20-ms speech frames. The interleaving structure is summarized in Figure 3.10

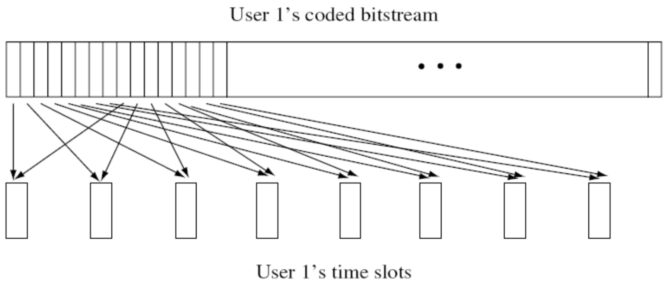


Figure 11: Rotation and repetition codes

- 456 bits are divided into eight blocks of 57 bits each. Different blocks here carry different bit positions. for example,
- 1st block contains bit numbers [0,8,16,...448],
- 2nd block contains bit numbers [1,9,17,...449],
- 3rd block contains bit numbers [2,10,18,...450],
- 4th block contains bit numbers [3,11,19,...451],
- 5th block contains bit numbers [4,12,20,...452],
- 6th block contains bit numbers [5,13,21,...453],
- 7th block contains bit numbers [6,14,22,...454],
- 8th block contains bit numbers [7,15,23,...455],
-

Antenna Diversity

Antenna Diversity

- Time diversity requires several **coherence time periods** (interleaving and coding).
- When there is a **strict delay constraint and/or the coherence time is large**, this may not be possible.
- The other form of diversity is **Antenna diversity or spatial diversity**.
- Antenna diversity can be obtained by placing **multiple antennas** at the transmitter and/or at the receiver.
- If the antennas are placed **far distance**, the channel gains between different antenna pairs fade more or less independently, and **independent signal paths are created**.
- The required antenna separation depends on the local **scattering environment** and **carrier frequency**.
- Typical antenna separation of **half to one carrier wavelength** is sufficient.
- In antenna diversity there are two types:
  1. **Receive diversity**
  2. **Transmit diversity**
- **Receive diversity:** using multiple receive antennas (single input multiple output or SIMO channels).
- **Transmit diversity:** using multiple transmit antennas (multiple input single output or MISO channels).

Receive diversity

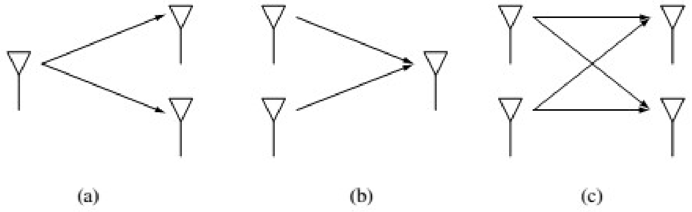


Figure 12: (a) Receive diversity (b) Transmit diversity (c) Transmit and receive diversity

- In a flat fading channel with 1 transmit antenna and L receive antennas the channel model is as follows:

$$y_\ell [m] = h_\ell [m] \, x [m] + w_\ell [m] \, , \ell = 1, ...L \tag{53}$$

  where the noise $w_\ell[m] \sim \mathcal{CN}(0, N_0)$ and is independent across the antennas.
- Detect $x [1]$ based on $y_1 [1] , ..., y_L [1]$ L diversity branches.
- This is similar to the repetition code and interleaving over time, with L diversity branches

- If the antennas are spaced sufficiently far apart, and assume that the gains $h\ell[1]$ are independent Rayleigh fading, and the diversity gain will be L.
- The error probability of BPSK conditional on the channel gains is:

$$Q\left(\sqrt{2\left\|h\right\|^{2}SNR}\right) \tag{54}$$

- while break up the total received SNR conditioned on the channel gains into a product of two terms:

$$\left\|h\right\|^{2}SNR = LSNR\frac{1}{L}\left\|h\right\|^{2} \tag{55}$$

- The first term corresponds to a power gain(also called array gain), by having multiple receive antennas and coherent combining at the receiver, the effective total received signal power increases linearly with L
- The second term reflects the diversity gain, when all the h_l are independent there is a diminishing marginal return as L increases:
- Due to the law of large numbers,the second term in

$$\frac{1}{L}\left\|h\right\|^{2} = \frac{1}{L}\sum_{l=1}^{L}\left|h_{l}\left[1\right]\right|^{2} \tag{56}$$

Transmit diversity: space-time codes

- Consider there are L transmit antennas and 1 receive antenna, the MISO channel.
- This is common in the downlink of a cellular system since it is often **cheaper** to have multiple antennas at the base-station than to have multiple antennas at every handset.
- Transmit the same symbol over the L different antennas during L symbol times to get a diversity gain of L.
- At any one time, only one antenna is turned on and the rest are silent.
- This is similar to a repetition code, repetition codes are quite wasteful of degrees of freedom.
- Use one antenna at a time and transmit the coded symbols of the time diversity code successively over the different antennas.
- **Alamouti scheme**, space-time code is used for the transmit diversity system.
- This is the transmit diversity scheme proposed in several third-generation cellular standards
- The Alamouti scheme is designed for **two transmit antennas**; generalization to more than two antennas is possible, to some extent.
- With flat fading, the two transmit, single receive channel is written as

$$y[m] = h_1[m]\,x_1[m] + h_2[m]\,x_2[m] + w[m]$$

where h_i is the channel gain from transmit antenna i.

- The Alamouti scheme transmits two complex symbols u_1 and u_2 over two symbol times.
- At time 1,

$$x_1[1] = u_1, \qquad x_2[1] = u_2$$

- At time 2,

$$x_1[2] = -u_2^*, \qquad x_2[2] = u_1^*$$

$$u = \begin{bmatrix} u_1 & -u_2^* \\ u_2 & u_1^* \end{bmatrix}$$

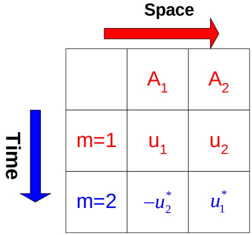


Figure 13: Space-time code

- Row 1 of u is transmitted on antenna 1. Row 2 of u is transmitted on antenna 2.
- Two time periods are needed to transmit u on a 2×1 MIMO channel.
- If the channel remains constant over the two symbol times then

$$\begin{aligned} h_1 &= h_1[1] = h_1[2] \\ h_2 &= h_2[1] = h_2[2] \end{aligned}$$

- Then in matrix form:

$$[y[1]\ y[2]] = [h_1 \quad h_2] \begin{bmatrix} u_1 & -u_2^* \\ u_2 & u_1^* \end{bmatrix} + [w[1]\ w[2]]$$

- In detecting $u_1,\ u_2$, rewrite equation as for column square matrix

$$\begin{bmatrix} y[1] \\ y[2]^* \end{bmatrix} = \begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} w[1] \\ w[2]^* \end{bmatrix}$$

- The columns of the square matrix are orthogonal.
- Project y onto each of the two columns to obtain the sufficient statistics

$$r_i = \|h\|\, u_i + w_i, i = 1, 2, ... \tag{57}$$

- where $h = [h_1,\ h_2]^t$
- and the noise $w_i \sim \mathcal{CN}(0, N_0)$ w_1, w_2 are independent.

The determinant criterion for space-time code design

- Consider a space-time code as a set of complex codewords X_i ,
- where each X_i is an L by N matrix. Here, L is the number of transmit antennas and N is the block length of the code.
- For example, in the Alamouti scheme, each codeword is of the form

$$\begin{bmatrix} u_1 & -u_2^* \\ u_2 & u_1^* \end{bmatrix} \tag{58}$$

- with L=2 and N=2.
- In each codeword in the **repetition scheme** is of the form

$$\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \tag{59}$$

- More generally, any block length L **time diversity** code with codewords X_i translates into a block length L transmit diversity code with codeword matrices X_i, where

$$X_i = diag\{x_{i1}, ..., x_{iL}\} \tag{60}$$

- Assuming that the channel remains constant for N symbol times, we can write

$$y^t = h^* X + w^t \tag{61}$$

where

$$y = \begin{bmatrix} y[1] \\ . \\ . \\ y[N] \end{bmatrix} \quad h = \begin{bmatrix} h_1^* \\ . \\ . \\ h_L^* \end{bmatrix} \quad w = \begin{bmatrix} w[1] \\ . \\ . \\ w[N] \end{bmatrix} \tag{62}$$

- To bound the error probability, consider the pairwise error probability of confusing X_B with X_A, when X_A is transmitted, conditioned on the fading gains h
- Deciding between the vectors $h * X_A$ and $h * X_B$ under additive circular symmetric white Gaussian noise.
- A sufficient statistic is $\mathcal{R}[v * y]$ where $v = h * (X_A - XB)$.
-
- The conditional pairwise error probability is

$$p\{X_A \to X_B|h\} = Q\left(\frac{||h^*(X_A - X_B)||}{2\sqrt{\frac{N_0}{2}}}\right) \tag{63}$$

Hence, the pairwise error probability averaged over the channel statistics is

$$p\{X_A \to X_B\} = E\left(Q\left(\sqrt{\frac{SNRh^*(X_A - X_B)(X_A - X_B)^*}{2}}\right)\right) \tag{64}$$

- $(X_A - X_B)(X_A - X_B)^*$ is Hermitian and is thus diagonalizable by a unitary transformation, $(X_A - X_B)(X_A - X_B)^* = U\Lambda U^*$ where U is unitary and $\Lambda = diag\{\lambda_1^2, ..., \lambda_L^2\}$
- Here λ_L are the singular values of the codeword difference matrix $X_A - X_B$.

- we can rewrite the pairwise error probability as

$$p\{X_A \to X_B\} = E\left(Q\left(\sqrt{\frac{SNR \sum_{l=1}^{L} \left|\widetilde{h}_l\right|^2 \lambda_\ell^2}{2}}\right)\right) \tag{65}$$

- where $\widetilde{h} = U^* h$

$$p\{X_A \to X_B\} \leq \prod_{l=1}^{L} \frac{1}{1 + SNR\frac{\lambda_l^2}{4}} \tag{66}$$

- If λ_l^2 are strictly positive for all the codeword differences, then the maximal diversity gain of L is achieved.
- If indeed all the λ_l^2 are positive, then,

$$p\{X_A \to X_B\} \leq \frac{4^L}{SNR^L \prod_{l=1}^{L} \lambda_l^2} \tag{67}$$

$$= \frac{4^L}{SNR^L \det\left[\left(\left(X_A - X_B\right)\left(X_A - X_B\right)^*\right)\right]} \tag{68}$$

and a diversity gain of L is achieved.

- The coding gain is determined by the minimum of the determinant over all codeword pairs.This is sometimes called the determinant criterion. $\det\left[\left(\left(X_A - X_B\right)\left(X_A - X_B\right)^*\right)\right]$

Frequency-Diversity

Frequency-Diversity

- The **coherence bandwidth** is defined as the frequency band within which all frequency components are **equally affected** by fading due to multipath propagation phenomena.
- Systems operating with channels substantially **narrower** than the coherence bandwidth are known as **Narrowband systems**.
- In Narrowband systems all the components of the signals are equally influenced by multipath propagation.
- Wideband systems operate with channels substantially **wider** than the **coherence bandwidth**.
- In wideband systems the various frequency components of the signal may be differently affected by fading.
- **Narrowband flat fading channels** are modeled by a single-tap filter, as most of the multipaths arrive during one symbol time.
- In **wideband channels**, the transmitted signal arrives over **multiple symbol times** and the multipaths can be resolved at the receiver.
- The frequency response is no longer flat, i.e., the transmission bandwidth W is greater than the coherence bandwidth W_c of the channel.
- This provides another form of diversity: **frequency diversity**.

- Consider a discrete-time baseband model of the wireless channel, the sampled output $y[m]$ is

$$y[m] = \sum_{\ell} h_\ell[m]x[m-\ell] + w[m] \tag{69}$$

- Where $h_\ell[m]$ denotes the ℓth channel filter tap at time m.
- Consider the situation when one symbol $x[0]$ is sent at time 0, and **no symbols** are transmitted after that. The receiver observes

$$y[\ell] = h_\ell[\ell]x\,[0] + w[\ell] \qquad \ell = 0, 1, 2....... \tag{70}$$

- The channel response has a finite number of taps L.
- The **delayed replicas of the signal** are providing L branches of diversity in detecting $x[0]$, since the tap gains $h_\ell[\ell]$ are assumed to be **independent**.
- This diversity is achieved by the ability of resolving the multipaths at the receiver and this is called *frequency diversity*.
- In this scheme, if we transmit symbols **more frequently**, inter-symbol interference (**ISI**) occurs: the delayed replicas of previous symbols **interfere** with the current symbol.
- To overcome ISI, there are three common approaches:

1. **Single-carrier systems with equalization**.
2. **Direct-sequence spread-spectrum**.
3. **Multi-carrier systems**.

Single-carrier with ISI equalization

- Single-carrier with ISI equalization is the classic approach to communication over frequency-selective channels.
- Starting at time 1, a sequence of uncoded independent symbols x[1], x[2],....is transmitted over the frequency-selective channel.
- Assuming that the channel taps **do not vary** over these N symbol times, the received symbol at time m is

$$y[m] = \sum_{\ell=0}^{L-1} h_\ell x[m-\ell] + w[m] \tag{71}$$

- The process of extracting the symbols from the received signal is called **equalization**.

Frequency-selective channel viewed as a MISO channel

- The frequency-selective channel is transformed into a flat fading MISO channel with L transmit antennas and a single receive antenna and channel gains h_0,.....$h_L - 1$.
- The transmission scheme on the MISO channel:
- At time 1, the symbol **x[1] is transmitted on antenna 1** and the other **antennas are silent**.
- At time 2, **x[1]** is transmitted at **antenna 2, x[2] is transmitted on antenna 1** and the other antennas remain **silent**.
- At time m, $x[m-\ell]$ is transmitted on antenna $\ell + 1$, for $\ell = 0, \ldots.., L-1$.

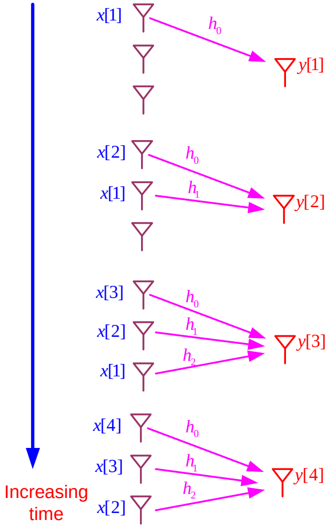


Figure 14: The MISO scenario equivalent to the frequency- selective channel.

- To achieve full diversity on a symbol $x[N]$, Observe the received symbols up to time N+L-1.
- Then the system in matrix form will be:

$$\mathbf{y^t} = \mathbf{h}^* \mathbf{X} + \mathbf{w^t}$$

- where $y^t = [y[1], ...y[N+L-1]]$ $h^* = [h_0, ..., h_L - 1]$ $w^t = [w[1], ..., w[N+L-1]]$
- The L by N+L-1 space-time code matrix

$$X = \begin{bmatrix} x[1] & x[2] & . & . & x[N] & . & . & x[N+L-1] \\ 0 & x[1] & x[2] & . & . & x[N] & . & x[N+L-2] \\ 0 & 0 & x[1] & x[2] & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 \quad 0 & 0 \quad 0 \; 0 & 0 \quad 0 & x[1] & x[2] & . & . & . \; . \quad x[N] \end{bmatrix} \quad (72)$$

- corresponds to the transmitted sequence

$$X = [x[1], ...x[N+L-1]]^t \quad (73)$$

Error probability analysis

- Consider the maximum likelihood detection of these sequence **x** based on the received vector **y** (MLSD).
- With MLSD, the pairwise error probability of confusing X_A with X_B, when X_A is transmitted is,

$$p\{X_A \rightarrow X_B\} \leq \prod_{\ell=1}^{L} \frac{1}{1 + SNR\lambda_l^2/4} \tag{74}$$

- where λ_l^2 are the eigenvalues of the matrix $(X_A - X_B)(X_A - X_B)^*$ and SNR is the total received SNR per received symbol.
- The error probability decays SNR^{-L} whenever the difference matrix X_A-X_B is of rank L.
- By a union bound argument, the probability of detecting the particular symbol X[N]incorrectly is bounded by

$$\sum_{X_B : x_B[N] \neq x_A[N]} p\{X_A \rightarrow X_B\} \tag{75}$$

Implementing MLSD: The Viterbi algorithm

- Given the received vector y of length n, MLSD requires solving the optimization problem

$$\max \mathbb{P}\{y|x\} \tag{76}$$

- Transmitted sequences are estimated by Viterbi algorithm.

- The memory in the frequency-selective channel can be represented by a finite state machine. At time m, define the state

$$S[m] = \begin{bmatrix} x[m-L+1] \\ x[m-L+2] \\ . \\ . \\ . \\ x[m] \end{bmatrix} \tag{77}$$

- An example of the finite state machine when the $x[m]$ are $\pm$ BPSK symbols.
- The number of states is M^L, where M is the constellation size for each symbol $x[m]$.

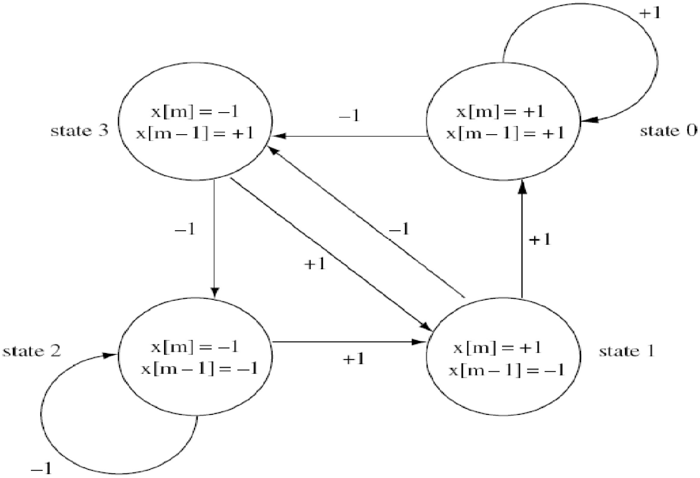


Figure 15: A finite state machine when x[m] are 1 BPSK symbols and L= 2.

- The received symbol y[m] is given by

$$y[m] = h^* s[m] + w[m] \tag{78}$$

- The MLSD problem can be rewritten as

$$\min_{s[1],\ldots\ldots,s[n]} -\log p\{y[1],\ldots\ldots y[n]\,|S[1],\ldots\ldots S[n]\} \tag{79}$$

- The received symbols are independent and the log-likelihood ratio breaks into a sum:

$$\log p\{y[1],\ldots\ldots y[n]\,|S[1],\ldots\ldots S[n]\} = \sum_{m=1}^{n} \log p\{y[m]\,|S[m]\} \tag{80}$$

- The optimization problem can be represented as the problem of finding the shortest path through an n-stage trellis, as shown in Figure 16.
- Each state sequence (s[1],....,s[n]) is visualized as a path through the trellis, and given the received sequence y[1],..., y[n].
- The cost associated with the mth transition is

$$C_m(s[m]) = -\log p\{y[m]\,|S[m]\} \tag{81}$$

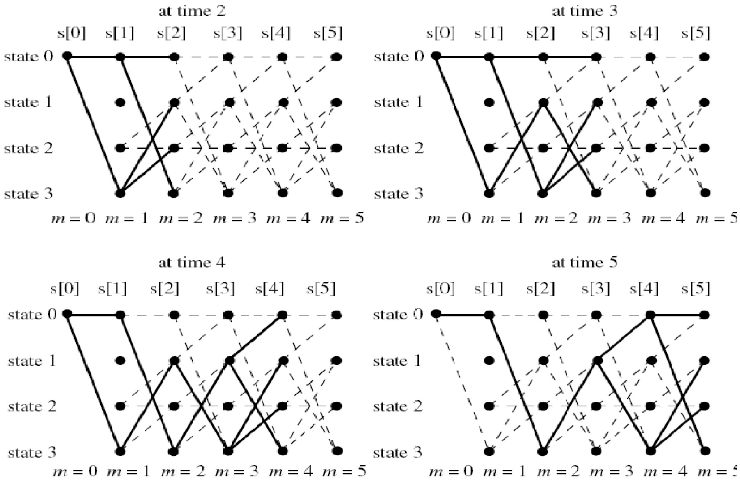


Figure 16: The trellis representation of the channel.

- Let $V_m(s)$ be the cost of the shortest path to a given state s at stage m.
- Then $V_m(s)$for all states s can be computed recursively.

$$V_1\left(s\right)=C_1\left(s\right)$$
$$V_m\left(s\right)=\min_u\left[V_{m-1}\left(u\right)+C_m\left[s\right]\right]$$

- u is the possible states that the finite state machine can be in at stage m-1.
- If the shortest path to state s at stage m goes through the state u^*. at stage m-1, then the part of the path up to stage m-1 must itself be the shortest path to state u_*.
- To compute the Shortest path up to stage m, it suffices to augment only the shortest paths up to stage m-1.

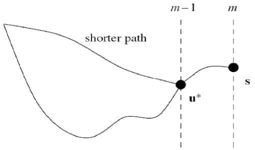


Figure 17: The dynamic programming principle.

- Once V_m(s) is computed for all states s, the shortest path to stage m is simply the minimum of these values over all states s.

Direct-sequence spread-spectrum

Direct-sequence spread-spectrum

- A common communication system that employs a wide bandwidth is the **direct-sequence (DS) spread-spectrum system** and its basic components are shown in Figure 18

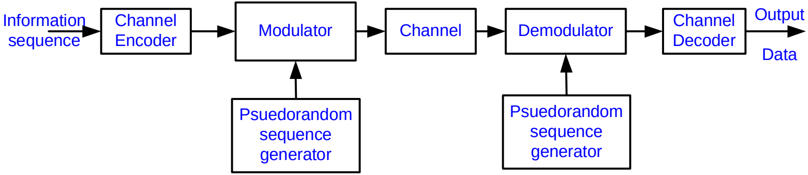


Figure 18: Basic elements of a direct sequence spread spectrum system

- Information is encoded and modulated by a pseudonoise(PN) sequence and transmitted over bandwidth W.
- The ratio W/R is called **processing gain**.
- In IS-95 (CDMA) is a direct-sequence spread-spectrum system the bandwidth is 1.2288MHz and a typical data rate (voice) is 9.6 kbits/s, so the processing gain is 128.
- Each sample period is called a chip, and the **chip rate is much larger** than the data rate.
- Because the symbol rate per user is very low in a spread-spectrum system, ISI is typically negligible and equalization is not required.

 Direct-sequence spread-spectrum

- Consider one of two n-chips long pseudonoise sequences X_A or X_B are transmit over a wideband multipath channel.
- The received signal is given by

$$y[m] = \sum_{\ell} h_{\ell}[m] x[m-\ell] + w[m] \tag{82}$$

- Assume that $h_\ell[m]$ does not vary with m during the transmission of the sequence. i.e., the channel is considered **time-invariant**.
- Also assume that there is negligible interference between consecutive symbols, so that we can consider the binary detection problem in isolation for each symbol.

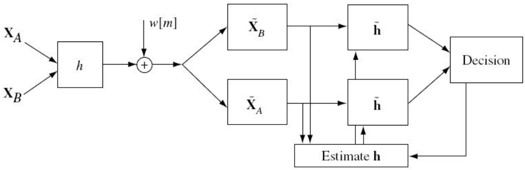


Figure 19: The Rake receiver

- Assume that simultaneously n>>T_dW and n<<T_cW, which is possible only if T_d<<T_c. In a typical cellular system, T_d is of the order of microseconds and T_c of the order of tens of milliseconds
- With the above assumptions, the output is a convolution of the input with the channel plus noise.

$$y[m] = (h * x)[m] + w[m] \tag{83}$$

- Assuming the channel h is known to the receiver, two sufficient statistics, r_A and r_B, can be obtained by projecting the received vector y=$[y[1], ...y[n+L]]^t$ onto the n+L dimensional vectors V_A and V_B, where

$$V_A = [(h * x_A)[1], ...(h * x_A)[n+L]]^t$$

$$V_B = [(h * x_B)[1], ...(h * x_B)[n+L]]^t$$

$$r_A = V_A^* Y$$

$$r_B = V_B^* Y$$

- The computation of r_A and r_B can be implemented by first matched filtering the received signal to x_A and to x_B.
- The outputs of the matched filters are passed through a filter matched to the channel response h and then sampled at time n+L (Figure 2). This is called the Rake receiver.
- What the Rake actually does is taking inner products of the received signal with shifted versions at the candidate transmitted sequences. Each output is then weighted by the channel tap gains at the appropriate delays and summed. The signal path associated with a particular delay is sometimes called a finger of the Rake receiver
- Assuming that the channel gains h_l are known at the receiver. These gains have to be estimated and tracked from either a pilot signal or in a decision-directed mode using the previously detected symbols.
- Due to hardware limitations, the actual number of fingers used in a Rake receiver may be less than the total number of taps L in the range of the delay spread.

Error probability analysis

- Consider the maximum likelihood detection of these sequence **x** based on the received vector **y** (MLSD).
- With MLSD, the pairwise error probability of confusing X_A with X_B, when X_A is transmitted is,

$$p\left\{X_A \rightarrow X_B\right\} \leq \prod_{\ell=1}^{L} \frac{1}{1 + SNR\lambda_l^2} \tag{84}$$

- where λ_l^2 are the eigenvalues of the matrix $\left(X_A - X_B\right)\left(X_A - X_B\right)^*$ and SNR is the total received SNR per received symbol.
- The error probability decays SNR^{-L} whenever the difference matrix X_A-X_B is of rank L.
- By a union bound argument, the probability of detecting the particular symbol X[N]incorrectly is bounded by

$$\sum_{X_B : x_B[N] \neq x_A[N]} p\left\{X_A \rightarrow X_B\right\} \tag{85}$$

Orthogonal Frequency Division Multiplexing

Orthogonal frequency division multiplexing

- The wireless channel constitutes a hostile propagation medium, which suffers from **fading** (caused by destructive addition of multipath components) and **interference** from other users.
- Diversity provides the receiver with several (ideally independent) **replicas** of the transmitted signal and is therefore a powerful means to combat **fading and interference** and thereby improve link reliability.
- The basic idea OFDM is to **divide** the available spectrum into several **subchannels** (subcarriers)
- OFDM is Based on the **Fast Fourier Transform**
- Standardized for Digital Audio Broadcasting (DAB), Digital Video Broadcasting (DVB), IEEE 802.11a, 802.16a, HyperLAN II
- Considered for **Fourth-Generation** (4G) mobile communication systems

- FDM channel is like water flow out of a **faucet**, whereas OFDM is like **shower**.
- In faucet water comes in one big stream whereas OFDM shower is made up of a lot of little streams
- FDM channel is like **big truck** carry a full load, whereas OFDM is like many **small truck** in which the load is distributed to each truck.
- In faucet case we can control the flow of water, whereas it is difficult to control.
- If **accident** occurs only small amount of data will suffer in OFDM.

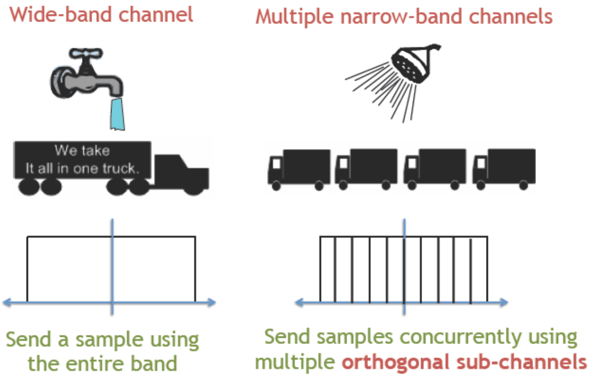


Figure 20: Basic concept of OFDM

Why OFDM is better?

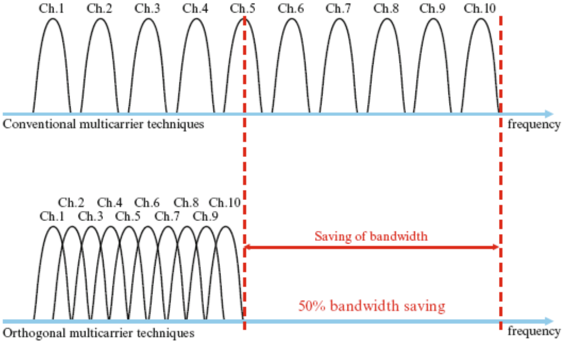


Figure 21: Concept of OFDM signal.

FDM requires guard bands between adjacent frequency bands extra overhead and lower throughput

OFDM does not requires guard bands between adjacent frequency bands because carriers are orthogonal to each other

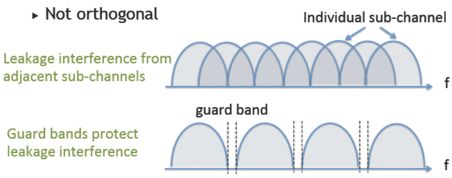


Figure 22: Concept of FDM signal.

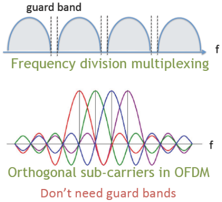


Figure 23: Concept of OFDM signal.

IFFT equation is

$$x(t) = \sum_{k=-N/2}^{N/2-1} X[k] e^{j2\pi kt/N}$$

FFT equation is

$$X[k] = \sum_{k=-N/2}^{N/2-1} x(t) e^{-j2\pi kt/N}$$

Orthogonal carriers referred to as subcarriers $f_i, i = 0, \ldots .N-1$.

$$x^{'}(t) = \sum_{k=0}^{N-1} x_R(t)cos(2\pi k f_o t) - x_I(t)sin(2\pi k f_o t)$$

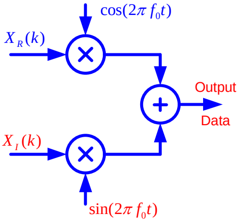


Figure 24: IFFT.

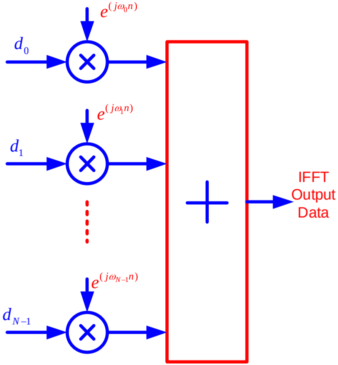


Figure 25: Basic concept of OFDM

Orthogonality of any two signals

$$d(t) = \sum_{t=N/2}^{N/2-1} e^{j2\pi kt/N} e^{j2\pi pt/N} = 0 \qquad p \neq k$$

- The above OFDM transmitter is mathematically equivalent to IFFT:

$$d(t) = \sum_{k=0}^{N-1} D[k] e^{j2\pi kn/N}$$

- In Orthogonal Frequency Division Multiplexing (OFDM) information is modulated on non-interfering sub-carriers in the frequency domain.
- If the channel is under spread (i.e., the coherence time is much larger than the delay spread) and is therefore approximately time-invariant for a sufficiently long time-scale, then transformation into the frequency domain can be a fruitful approach to communication over frequency-selective channels.
- This is the basic idea behind OFDM.
- Consider a discrete-time baseband model

$$y[m] = \sum_{\ell} h_\ell[m]x[m-\ell] + w[m]$$

- Assuming that the channel is **linear time invariant** then the channel model is

$$y[m] = \sum_{\ell=0}^{L-1} h_\ell x[m-\ell] + w[m]$$

Cyclic Prefix

- In multipath channel, **delayed replicas of previous** OFDM signal lead to ISI between successive OFDM signals.
- Solution : Insert a **guard interval** between successive OFDM signals
- A **cyclic prefix** is a **copy of the last part** of the symbol which is prepended to the transmitted symbol as shown in figure.
- This makes the transmitted signal periodic which plays a decisive roll in avoiding intersymbol and intercarrier interference

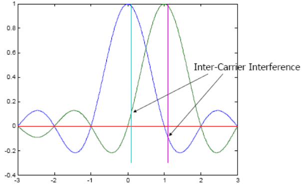


Figure 26: Inter-Carrier Interference (ICI)

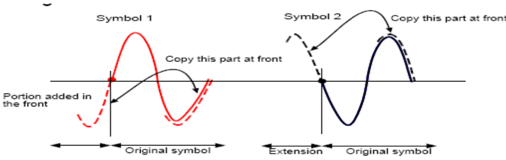


Figure 27: Cyclic prefix

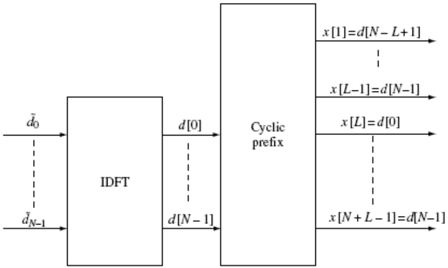


Figure 28: The OFDM transmission and reception scheme.

- For every block of symbols of length Nc, denoted by

$$d = [d[0], d[1], ..., d[N_c - 1]]^t \tag{86}$$

- Create an $N_c - L + 1$ input block as

$$x = [d[N_C - L + 1], d[N_C - L + 2], ...d[N_C - 1], d[0], d[1], ...d[[N_C - 1]]^t$$

- Add a prefix of length $L - 1$ consisting of data rotated cyclically symbols

- Consider the output

$$y[m] = \sum_{\ell=0}^{L-1} h_\ell x[m-\ell] + w[m]$$

- Due to additional cyclic prefix,the output over this time interval is

$$y[m] = \sum_{\ell=0}^{L-1} h_\ell d[(m-L-\ell) \bmod N_C] + w[m] \quad (87)$$

- Denoting the output of length N_c by

$$y = [y[L], ...y[N_c + L - 1]]^t$$

- the channel by a vector of length Nc

$$h = [h_0, h_1, ..., h_{L-1}, 0, ..., 0]^t$$

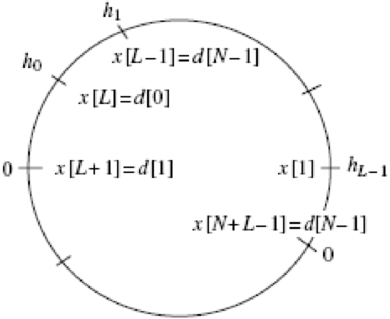


Figure 29: Convolution of h and the input x

- Equation 87 can be written as

$$y = h \otimes d + w \qquad (88)$$

- Taking the fourier transform of both sides of equation 87 and using the identity

$$DFT(h \oplus d)_n = \sqrt{N_c} DFT(h)_n DFT(d)_n$$

and (10) can be written as

$$\tilde{y}_n = \tilde{h}_n \tilde{d}_n + \tilde{w}_n$$

- Here $\tilde{w}_0........\tilde{w}_N c-1$ is the N_c point DFT of the noise vector w[1],......w[N_c]. The vector $[\tilde{h}_0,........\tilde{h}_N c-1]$ is defined as the DFT of the L tap channel h,multiplied by $\sqrt{N_c}$

$$\tilde{h}_n = \sum_{l=0}^{L-1} h_l \exp\left(\frac{-j2\pi nl}{N_C}\right)$$

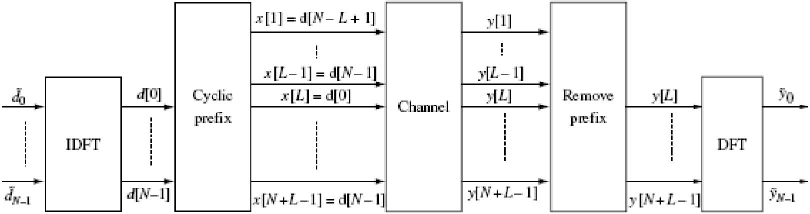


Figure 30: The OFDM transmission and reception scheme.

The circular convolution operation $u = h \otimes d$ can be viewed as a linear transformation

$$\mathbf{u} = \mathbf{C}\mathbf{d}$$

where

$$\mathbf{C} := \begin{bmatrix} h_0 & 0 & . & 0 & h_{L-1} & h_{L-2} & . & h_1 \\ h_1 & h_0 & . & 0 & h_{L-1} & h_{L-2} & . & 0 \\ . & . & . & . & . & . & . & . \\ 0 & . & 0 & h_{L-1} & h_{L-2} & . & h_1 & h_0 \end{bmatrix}$$

is a circulant matrix, i.e., the rows are cyclic shifts of each other.

References

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Y. Q. Shi, X. M. Zhang, Z.-C. Ni, and N. Ansari, "Interleaving for combating bursts of errors," *IEEE Circuits And Systems Magazine*, pp. 29–42, 2004.