Solutions to problems from Chapter 3

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For a systematic (7,4) linear block code, the parity matrix P is given by

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

i)Find all possible code vectors. ii)Draw the corresponding encoding circuit iii)Detect and correct the following error R=[1 0 1 1 1 0 0] iv)Draw the syndrome calculation circuit

Solution:

n=7 and k=4

There are $2^4 = 16$ message vectors given by u=[0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111]

The code vectors V are estimated using generator matrix G which is in the form of

$$G = [P|I] = \begin{bmatrix} 1 & 1 & 1| & 1 & 0 & 0 & 0 \\ 1 & 1 & 0| & 0 & 1 & 0 & 0 \\ 1 & 0 & 1| & 0 & 0 & 1 & 0 \\ 0 & 1 & 1| & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$V = (u_0, u_1, u_2, u_3) \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$v_6 = u_3,\ v_5 = u_2,\ v_4 = u_1,\ v_3 = u_0$

$v_2 = u_0 + u_2 + u_3,\ v_1 = u_0 + u_1 + u_3,\ v_0 = u_0 + u_1 + u_2$

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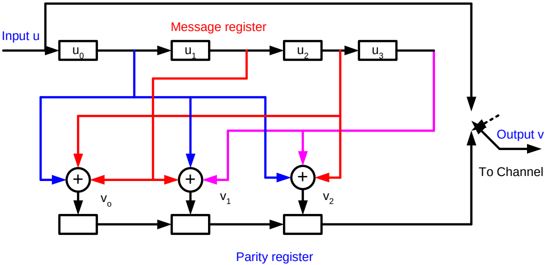
	Message	Code vector		Message	Code vector
1	0000	0000000	9	0001	0110001
2	1000	1111000	10	1001	1001001
3	0100	1100100	11	0101	1010101
4	1100	0011100	12	1101	0001101
5	0010	1010010	13	0011	0100011
6	1010	0101010	14	1011	0111011
7	0110	0110110	15	0111	0000111
8	1110	1001110	16	1111	1111111

Encoder circuit



Syndrome circuit:

$$H = [I_{n-k} | P^T] = \begin{bmatrix} 1 & 0 & 0| & 1 & 1 & 1 & 0 \\ 0 & 1 & 0| & 1 & 1 & 0 & 1 \\ 0 & 0 & 1| & 1 & 0 & 1 & 1 \end{bmatrix}$$

Syndrome vector is obtained from the received sequence $r = [r_0 \ \ r_1 \ldots r_7]$ using the parity check matrix of the code:

$$s = [s_0 \ s_1 \ s_2] = rH^T :$$

$$= (r_0, \ r_1, \ r_2, \ r_3, \ r_4, \ r_5, \ r_6) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

By parity check matrix the syndrome digits are:

$s_0 = r_0 + r_3 + r_4 + r_5$ $s_1 = r_1 + r_3 + r_4 + r_6$ $s_2 = r_2 + r_3 + r_5 + r_6$

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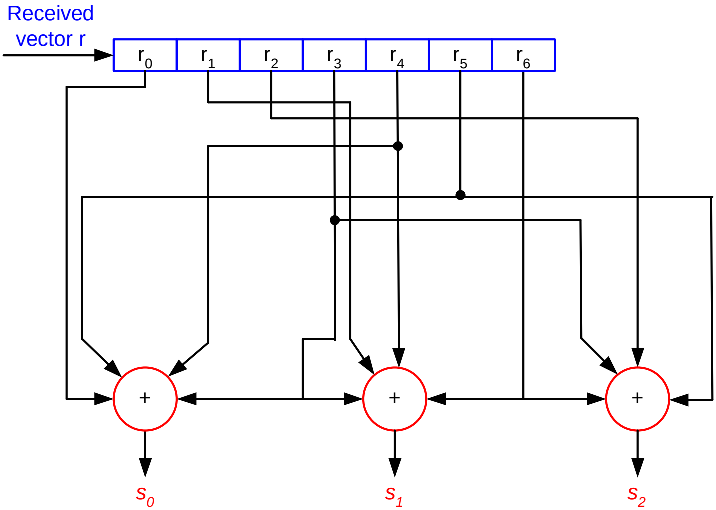

Figure: Syndrome calculation circuit

$$s = [s_0\ s_1\ s_2] = rH^T:$$

$$= (r_0,\ r_1,\ r_2,\ r_3,\ r_4,\ r_5,\ r_6)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= (1,\ 0,\ 1,\ 1,\ 1,\ 0,\ 0)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$s = [1\ 0\ 0]$$

Consider a systematic (8, 4) code whose parity-check equations are

$$v_0 = u_1 + u_2 + u_3$$
$$v_1 = u_0 + u_1 + u_2$$
$$v_2 = u_0 + u_1 + u_3$$
$$v_3 = u_0 + u_2 + u_3$$

where u_0, u_1, u_2, u_3 are message digits and $v_0\ v_1\ v_2\ v_3$ are parity check digits. Find the generator and parity check matrices for this code. Show analytically that the minimum distance of this code is 4.

Solution:

The information sequences $u = [u_0\ u_1\ u_2\ u_3]$ are encoded into codewords v of length n = 8, using a systematic encoder. If we assume that the systematic positions in a codeword are the last k = 4 positions, then codewords have the form $v = [v_0\ v_1\ v_2\ v_3\ u_0\ u_1\ u_2\ u_3]$ and the systematic encoding is specified by

$v = uG; G = [P \mid I]$:

$$G = [P|I] = \begin{bmatrix} 0 & 1 & 1 & 1| & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0| & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1| & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1| & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H = [I_{n-k}|P^T] = \begin{bmatrix} 1 & 0 & 0 & 0| & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0| & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0| & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1| & 1 & 0 & 1 & 1 \end{bmatrix}$$

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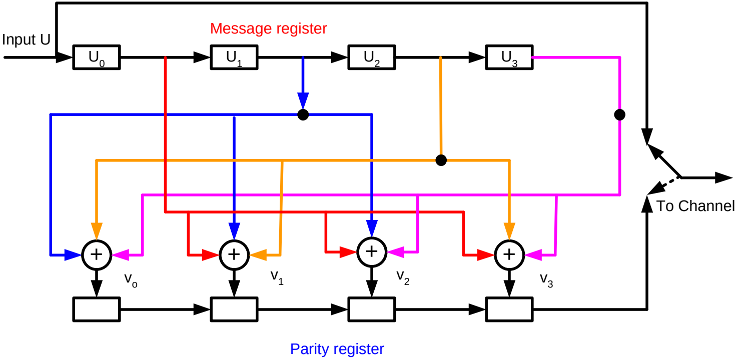
- To find the minimum distance analytically, we use the property that the minimum distance of a binary linear code is equal to the smallest number of columns of the parity-check matrix H that sum up to zero. Hence, we find that:
- there are no two identical columns in H $d_{min} > 2$
- there are no groups of 3 columns that sum up to 0 $d_{min} > 3$
- there exists a group of 4 columns (for example, columns 1,2,3,6) that sum up to 0 $d_{min} = 4$.

- 3.2 Construct an encoder for the code given in Problem 3.1
  Solution:
- Based on the parity check equations from the previous problem, the systematic encoder for the (8; 4) code has the structure as shown in Figure 1.

$$v_0 = u_1 + u_2 + u_3 \qquad v_1 = u_0 + u_1 + u_2$$
$$v_2 = u_0 + u_1 + u_3 \qquad v_3 = u_0 + u_2 + u_3$$



- 3.3 Construct a syndrome circuit for the code given in Problem 3.1
  Solution:
- Syndrome vector is obtained from the received sequence $r = [r_0 \;\; r_1 \ldots r_7]$ using the parity check matrix of the code:

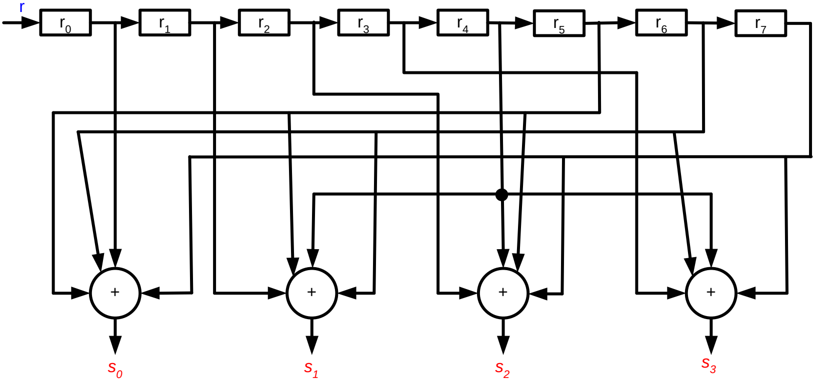
$$s = [s_0 \; s_1 \; s_2 \; s_3] = rH^T :$$

$$= (r_0, \; r_1, \; r_2, \; r_3, \; r_4, \; r_5, \; r_6, \; r_7) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

- By parity check matrix the syndrome digits are:

$$s_0 = r_0 + r_5 + r_6 + r_7 \qquad s_1 = r_1 + r_4 + r_5 + r_6$$
$$s_2 = r_2 + r_4 + r_5 + r_7 \qquad s_3 = r_3 + r_4 + r_6 + r_7$$

$$s_0 = r_0 + r_5 + r_6 + r_7 \qquad s_1 = r_1 + r_4 + r_5 + r_6$$
$$s_2 = r_2 + r_4 + r_5 + r_7 \qquad s_3 = r_3 + r_4 + r_6 + r_7:$$



The parity check bits of a (7,4) Hamming code, are generated by

$$\begin{aligned} c_5 &= d_1 + d_3 + d_4 \\ c_6 &= d_1 + d_2 + d_3 \\ c_7 &= d_2 + d_3 + d_4 \end{aligned}$$

i) Find the generator matrix [G] and parity check matrix [H] for this code ii) Prove that $GH^T = 0$ iii) The (n,k) linea block code so obtained has a 'dual' code. This dual code is a (n, n-k) code having a generator matrix H and parity check matrix G. Determine the eight code vectors of the dual code for the (7,4) Hamming code describe above. iv) Find the minimum distance of the dual code determined in part (c).

Solution:
n=7 and k=4
It is a 4×7 matrix in which 4×4 identity matrix
Message bits are d_1, d_2, d_3, d_4 and code bits are arranged as $c_1 = d_1$, $c_2 = d_2$, $c_3 = d_3$, $c_4 = d_4$, $c_5 = d_1 + d_3 + d_4$ $c_6 = d_1 + d_2 + d_3$, $c_7 = d_2 + d_3 + d_4$,

The parity check matrix H is $H = [P^T | I_{n-k}] = [P^T | I_3]$

$$G = [I|P] = \begin{bmatrix} 1 & 0 & 0 & 0| & 1 & 1 & 0 \\ 0 & 1 & 0 & 0| & 0 & 1 & 1 \\ 0 & 0 & 1 & 0| & 1 & 1 & 1 \\ 0 & 0 & 0 & 1| & 1 & 0 & 1 \end{bmatrix} \quad H = [P^T | I_3] = \begin{bmatrix} 1 & 0 & 1 & 0| & 1 & 0 & 0 \\ 1 & 1 & 1 & 0| & 0 & 1 & 0 \\ 0 & 1 & 1 & 1| & 0 & 0 & 1 \end{bmatrix}$$

ii)
The parity check matrix H is $H = [P^T | I_{n-k}] = [P^T | I_3]$

$$[G][H^T] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The dual code is $n, n-k = 7, 3$ with message bits as $d_1^{'}, d_2^{'}, d_3^{'}$

$$[c] = [d_1^{'}, d_2^{'}, d_3^{'}][H] = [d_1^{'}, d_2^{'}, d_3^{'}] \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

	Message	Code vector
1	000	0000000
2	001	0111001
3	010	1110010
4	011	1001011
5	100	1011100
6	101	1100101
7	110	0101110
8	111	0010111

By observing the dual code vector the minimum distance minimum number of ones it is four Hence $d_{min} = H_{min} = 4$

Find the generator G and parity check matrix H for a linear block code with matrix H for a linear block code with minimum distance three and message block size of eight bits bits.

Solution:

The code length n and k are related by $n \leq 2^{n-k} - 1$. The value of k given is k=8 then $n \leq 2^{n-8} - 1$, this equation will be satisfied for n=12, hence it is (12, 8)code.

The transpose of H matrix is given by $H = \begin{bmatrix} P_{kx(n-k)} \\ -- \\ I_{(n-k)} \end{bmatrix} = \begin{bmatrix} P_{8x4} \\ -- \\ I_4 \end{bmatrix}$

There are $2^4 = 16$ combinations which are 0000 0001 0010 0011 0100 0101 0110 0111 1000 1001 1010 1011 1100 1101 1110 1111 in which 0000 is not row, 0001 0010 0100 1000 are the rows of identity matrix, remaining codes are selected for the row arrangement for the H

$$H^T = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H = [P^T | I_4] = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1| & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1| & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1| & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0| & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$G = [I_8 | P_{8X4}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0| & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0| & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0| & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0| & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0| & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0| & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0| & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1| & 1 & 1 & 0 & 0 \end{bmatrix}$$

For a systematic (6,3) linear block code, the parity matrix P is given by

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

i)Find all possible code vectors ii) Construct the standard array table for the code words. iii) If the received the bit pattern r=[1 0 1 1 0 0] determine the syndrome , correctable error pattern and corrected code vector for a single bit error.

Solution:

n=6 and k=3

There are $2^3 = 8$ message vectors given by u=[000, 100, 010, 110, 001, 101, 011, 111]

The code vectors V are estimated using generator matrix G which is in the form of

Message bits are u_0, u_1, u_2 and code bits are arranged as $v_0, v_1, v_2, v_3, v_4, v_5$

$$G = [P|I] = \begin{bmatrix} 1 & 0 & 1| & 1 & 0 & 0 \\ 0 & 1 & 1| & 0 & 1 & 0 \\ 1 & 1 & 0| & 0 & 0 & 1 \end{bmatrix}$$

$$V = (u_0, u_1, u_2) \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

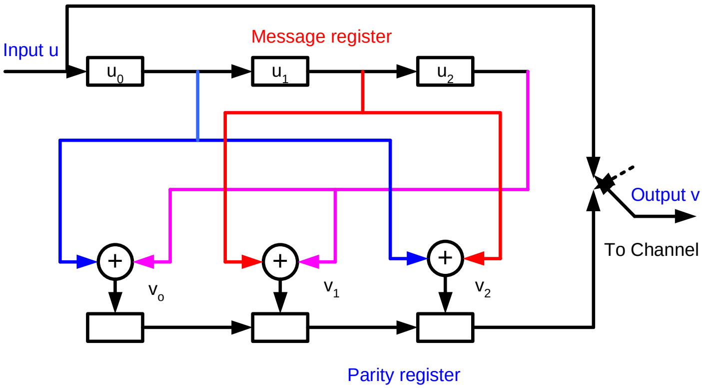
$v_5 = u_2, \; v_4 = u_1, \; v_3 = u_0$

$v_2 = u_0 + u_1, \; v_1 = u_1 + u_2, \; v_0 = u_0 + u_2$

When message is 1 0 1 then 0 1 1 1 0 1

When message is 0 1 1 then 1 0 1 0 1 1

	Message	Code vector
1	000	000000
2	100	101100
3	010	011010
4	110	110110
5	001	110001
6	101	011101
7	011	101011
8	111	000111



Input u
Message register
u0
u1
u2
Output v
To Channel
+
+
+
vo
v1
v2
Parity register

3.3 Determine the weight distribution of the (8,4) linear code given in Problem 3.1. Let the transition probability of a BSC be $\rho = 10^{-2}$. Compute the probability of an undetected error of this code.

Solution:

- Let A_i denote the number of codewords of weight i in an (n; k) code C. Then the numbers $A_0, A_1, \ldots An$ are called the weight distribution of the code. For any linear code A0 = 1 (every linear code must contain the all-zero codeword). The first next non-zero element of the weight distribution of a linear code is $A_{d_{min}}$, corresponding to the number of minimal-weight codewords.
- For the (8; 4) code with the minimum distance $d_{min} = 4$, all the codewords except the all-zero and the all-one codeword have minimum weight, that is, the weight distribution is $A_0 = 1, A_4 = 14, A_8 = 1,$

$A_0 = 1, A_4 = 14, A_8 = 1,$

- The probability of an undetected error is

$$P_u(E) = \sum_{i=1}^{n} A_i p^i (1-p)^{n-i}$$

$$P_u(E) = A_4 p^4 (1-p)^4 + A_8 p^8 = 14p^4(1-p^4) + p^8$$

- If $p = 10^{-2}$ then $P_u(E) = 1.3448x10^{-7}$.

- Form the generator matrix of the first-order RM code RM(1,3) of length 8. What is the minimum distance of the code? Determine its parity check sums and devise majority-logic decoder for the code. Decode the received vector r=(0 1 0 0 0 1 0 1).
  Solution:
- Consider the code R(1,3) with generator matrix:
- Code length: $n = 2^m = 2^3 = 8$
- $k(r, m) = 1 + \left(\begin{array}{c} 3 \\ 1 \end{array} \right) = 1 + \frac{3.2.1}{(2.1).1} = 4$
- Minimum distance: $d_{min} = 2^{m-r} = 2^{3-1} = 4$

$$\mathrm{G_{RM}}(1,3) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

- Form the generator matrix of the first-order RM code RM(1,3) of length 8. What is the minimum distance of the code? Determine its parity check sums and devise majority-logic decoder for the code. Decode the received vector r=(0 1 0 0 0 1 0 1).
  Solution:
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- Minimum distance: $d_{min} = 2^{m-r} = 2^{3-1} = 4$

$$\mathrm{G_{RM}}(1,3) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$